BENDING, ENTROPY AND PROPER AFFINE ACTIONS OF SURFACE GROUPS

MARTIN BRIDGEMAN, RICHARD CANARY, AND ANDRES SAMBARINO

Abstract. We show that for any closed surface S there is an explicit neighborhood V of the fuchsian locus in quasifuchsian space $\mathrm{QF}(S)$ such that for every representation $\rho \in V$ which is not fuchsian, there is a proper affine action on $\mathfrak{sl}(2,\mathbb{C})$ with linear part $\mathrm{Ad}(\rho)$. We further show that there is a larger neighborhood U of the Fuchsian locus so that every critical point of the entropy function in U lies on the Fuchsian locus.

Contents

1. Introduction

Quasifuchsian hyperbolic 3-manifolds are a central object of study in low-dimensional topology and dynamics. One classical theme is the relationship between the geometry of the convex core and dynamical quantities like the topological entropy of the geodesic flow. More recently, connections have emerged between quasifuchsian manifolds and proper affine actions of surface groups on $\mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{R}^6$. In this paper, we investigate bending deformations on the space $QF(S)$ of all (marked) quasifuchsian hyperbolic 3-manifolds. We introduce the notion of a moderately bent Jordan domain and obtain applications to both the variation of the entropy function on $QF(S)$ and to proper affine actions of surface groups.

We begin our investigation by using work Kourouniotis [46, 47] to establish a general formula for the variation of the complex length function of an element of $\pi_1(S)$ during a bending deformation. If a boundary component of the convex core is moderately bent, we obtain control on the real part of the variation as we bend along the associated bending lamination.

Our first application of this work is to show that if one component of the boundary of the convex core of a quasifuchsian hyperbolic 3-manifold (which is not fuchsian) is moderately bent, then it is not a critcal point of the entropy function on $QF(S)$. Recall that the topological entropy of the geodesic flow of a quasifuchsian hyperbolic 3-manifold is the exponential growth rate of the number of closed geodesics of length at most T. The topological entropy is an analytic function on $\mathrm{QF}(S)$ (see Ruelle [57]) and it achieves its minimum value, which is 1, exactly along the locus of fuchsian groups (see Bowen [11]).

Boston College
University of Michigan
CNRS - Université Sorbonne Paris Nord
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Previous work of Bridgeman [12] provided some evidence that every critical point of the entropy function lies on the Fuchsian locus.

Our second application is to show that if $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ is the holonomy representation of a quasifuchsian hyperbolic 3-manifold (which is not fuchsian) and both components of the boundary of the convex core are moderately bent, then $\mathrm{Ad}\,\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$. Danciger, Guéritaud and Kassel [22] previously exhibited specific quasifuchsian hyperbolic 3-manifolds with this property. We extend our results to find proper affine actions of surface groups on any simple complex Lie group. We also give applications to proper actions of surface groups on the group manifold of a complex simple Lie group.

Finally, we use work of Bridgeman, Canary and Yarmola [15] to find explicit bounds on the "roundness" of the bending lamination which guarantee that the associated boundary component of the convex core is moderately bent. We can then describe an explicit open neighborhood U of the Fuchsian locus in $QF(S)$ so that every manifold in U which is not fuchsian is not a critical point of the entropy function. Similarly, we describe an explicit open neighborhood V of the Fuchsian locus in $QF(S)$ so that if ρ is the holonomy representation of manifold in V which is not fuchsian, then $\mathrm{Ad}\,\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$.

We next develop the language to state our results more concretely. We follow this with a more in-depth discussion of the history of proper affine actions.

Statement of results: A *quasifuchsian* representation is a discrete faithful representation $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$, where S is a closed oriented surface of genus at least two, such that the limit set $\Lambda(\rho)$ of $\rho(\pi_1(S))$ is a Jordan curve. *Quasifuchsian space* $\mathrm{QF}(S)$ is the space of conjugacy classes of quasifuchsian representations. We may regard $\mathrm{QF}(S)$ as an open subset of the character variety

$$\mathfrak{X}(S) = \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))//\mathsf{PSL}(2,\mathbb{C}).$$

Since every point in $QF(S)$ is a smooth point of $\mathfrak{X}(S)$ (see [36]), $QF(S)$ has the structure of a complex manifold.

If ρ is quasifuchsian then the complement of the limit set $\Lambda(\rho)$ is two Jordan domains $\Omega_+(\rho)$ and $\Omega_-(\rho)$. The group $\rho(\pi_1(S))$ acts properly discontinuously on $\hat{\mathbb{C}} - \Lambda(\rho)$, so one obtains two marked Riemann surfaces $X_+(\rho) = \Omega_+(\rho)/\rho(\pi_1(S))$ and $X_-(\rho) = \Omega_-(\rho)/\rho(\pi_1(S))$ where we choose the labels on the components of $\hat{\mathbb{C}} - \Lambda(\rho)$ so that there is an orientation-preserving homeomorphism from $S \cup \bar{S} \to X_+(\rho) \cup X_-(\rho)$ in the homotopy class determined by ρ (where $\bar{S}$ is S with the opposite orientation). Bers showed that the resulting map from $\mathrm{QF}(S)$ to $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$ is a real analytic diffeomorphism, where $\mathcal{T}(S)$ is the Teichmüller space of S. A representation $\rho \in \mathrm{QF}(S)$ is *fuchsian* if it is conjugate to a representation with image in $\mathsf{PSL}(2,\mathbb{R})$. The *Fuchsian locus* $\mathrm{F}(S) \subset \mathrm{QF}(S)$ of fuchsian representations is identified with the diagonal in this parametrization.

If Ω is a Jordan domain whose boundary is a Jordan curve C, we say a pair (x, y) of distinct points in C is a *bending pair* for Ω if there is a maximal round open disk D in Ω such that $x, y \in \partial D$. We say that Ω is *moderately bent* if for every bending pair (x, y) there exists a circle L transverse to C such that $L \cap C = \{x, y\}$.

We define the *entropy function* $h : \mathrm{QF}(S) \to [1, 2)$ by letting $h([\rho])$ be the topological entropy of the geodesic flow on $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$, i.e. the exponential growth rate of the number of closed geodesics in N_ρ of length at most T. Sullivan [62] proved that $h([\rho])$ is the Hausdorff dimension of the limit set $\Lambda(\rho)$ of $\rho(\pi_1(S))$. One may also define $h([\rho])$ to be the *critical exponent* of the *Poincaré series*

$$P_\rho(s) = \sum_{\gamma \in \pi_1(S)} e^{-sd(x_0, \gamma(x_0))},$$

for any $x_0 \in \mathbb{H}^3$, i.e. $P_\rho(s)$ converges if $s > h([\rho])$ and diverges if $s < h([\rho])$.

Ruelle [57] showed that h is real analytic. Bowen [11] showed that h attains its minimum value exactly along the Fuchsian locus. One might hope that h has no other critical points. Bridgeman [12] showed that the Hessian of the entropy function is positive definite on at least a half-dimensional subspace, so the entropy functional has no local maxima.

We show that if one of the components of the complement of the limit set of a quasifuchsian (but not fuchsian) representation ρ is moderately bent, then ρ is not a critical point of the entropy function.

Theorem 1.1. *Suppose that $[\rho] \in \mathrm{QF}(S)$ is not fuchsian and either $\Omega_+(\rho)$ or $\Omega_-(\rho)$ is moderately bent, then $[\rho]$ is not a critical point of the entropy function h.*

We recall that if V is a finite-dimensional vector space, then the set $\mathrm{Aff}(V)$ of affine transformations of V is the semi-direct product of $\mathsf{GL}(V) \ltimes V$. The action of $(A, w) \in \mathrm{Aff}(V)$ is given by

$$(A, w)(v) = Av + w \quad \text{for all} \quad v \in V.$$

Let $\mathrm{Ad} : \mathsf{PSL}(2, \mathbb{C}) \to \mathsf{SL}(\mathfrak{sl}(2, \mathbb{C})) \subset \mathrm{Aff}(\mathfrak{sl}(2, \mathbb{C}))$ be the adjoint representation, i.e. $\mathrm{Ad}(A)(v) = AvA^{-1}$ for all $v \in \mathfrak{sl}(2, \mathbb{C})$. If $\sigma : \Gamma \to \mathrm{Aff}(V)$ is a representation, then the restriction of σ to the first factor $\mathsf{GL}(V)$ is a representation which we call the *linear part* of σ.

Theorem 1.2. *If $[\rho] \in \mathrm{QF}(S)$ is not fuchsian and both $\Omega_+(\rho)$ and $\Omega_-(\rho)$ are moderately bent, then there exists a representation $\sigma : \pi_1(S) \to \mathrm{Aff}(\mathfrak{sl}(2, \mathbb{C}))$ whose linear part is $\mathrm{Ad}(\rho)$ so that $\sigma(\pi_1(S))$ acts properly discontinuously on $\mathfrak{sl}(2, \mathbb{C})$.*

The proofs of both Theorem 1.1 and Theorem 1.2 involve studying bending deformations of quasifuchsian representations. If ρ is quasifuchsian, then its *convex core* $C(\rho)$ is the quotient of the convex hull $CH(\rho)$ of $\Lambda(\rho)$ in $\mathbb{H}^3$ by $\rho(\pi_1(S))$. If ρ is not fuchsian, then $C(\rho)$ is homeomorphic to $S \times [0, 1]$ and has two boundary components $\partial C_\pm(\rho)$, each of which is totally geodesic in the complement of a geodesic lamination $\beta_\pm$. We choose our signs so that $\partial CH_+(\rho)$ faces $\Omega_+(\rho)$. We often abuse notation by saying that $\partial C_\nu(\rho)$ is moderately bent when $\Omega_\nu(\rho)$ is moderately bent. The intrinsic metric on each component of $\partial C(\rho)$ is hyperbolic, and one may associate a transverse measure to $\beta_\pm$ which measures the total bending of $\partial C_\pm(\rho)$ along the lamination, to obtain the *bending laminations* (see Epstein-Marden [27] for background on convex hulls). Dular and Schlenker [26] recently showed that ρ is completely determined by its pair of bending laminations.

If ρ is quasifuchsian and λ is a measured lamination, one may define a *bending deformation* $\{\rho_{z\lambda}\}_{z\in\mathbb{C}} \subset \mathfrak{X}(S)$ of ρ along λ. If λ is a simple closed separating curve a, then we may write $\pi_1(S) = \pi_1(S_1) *_{\langle\alpha\rangle} \pi_1(S_2)$ and define $\rho_{z\lambda}$ for any $z \in \mathbb{C}$ by letting $\rho_{z\lambda}|_{\pi_1(S_1)} = \rho|_{\pi_1(S_1)}$ and $\rho_{z\lambda}(g) = A_z\rho(g)A_z^{-1}$ for all $g \in \pi_1(S_2)$ where A_z has the same axis as $\rho(g)$ and complex translation length z. In this convention, pure twisting corresponds to the case when z is totally real and pure bending corresponds to the case where z is totally imaginary. We generalize work of Kourouniotis [46, 47] to obtain a formula for the derivative of the complex length of any element of $\pi_1(S)$, see Theorem 3.2. The bending deformation is the complex Hamiltonian of the complex length function $\mathcal{L}_\lambda : \mathrm{QF}(S) \to \mathbb{C}$ of the lamination λ with respect to the natural complex symplectic structure on $\mathrm{QF}(S)$ described by Goldman (see [37, 36]), see Platis [56].

We generalize work of Kourouniotis [46, 47] to obtain a formula for the derivative of the complex length of an element $\gamma \in \pi_1(S)$ with respect to a bending deformation along a bending lamination β_ν for $\rho \in QF(S)$, see Theorem 3.2. Our formula is expressed in terms of the complex distances between the axis of $\rho(\gamma)$ and lifts of leaves of β_n. Assuming that $\Omega_\nu(\rho)$ is moderately bent places restrictions on these complex distances.

If ρ is quasifuchsian and β_ν is a bending lamination of ρ (where $\nu \in \{\pm\}$) and $\{\rho_t\}$ is the bending deformation of ρ_0 along $-i\beta_\nu$, then we define the *infinitesmal bending deformation* along $-i\beta_\nu$ to be

$$w_\nu(\rho) = \frac{d}{dt}\Big|_{t=0} \rho_{-ti\beta_\nu} \in \mathsf{T}_{[\rho]}\,\mathrm{QF}(S).$$

(As defined, $w_\nu(\rho)$ lies in $\mathsf{T}_\rho \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2, \mathbb{C}))$, but it descends to an element of $\mathsf{T}_{[\rho]}\,\mathrm{QF}(S)$ which is independent of the choice of representative of $[\rho]$.) We choose $-it\beta_\nu$ as our deformation since this corresponds to increasing the bending measure of the bending lamination on the convex core boundary. If $\gamma \in \pi_1(S)$, consider the real analytic function $\ell_\gamma : \mathrm{QF}(S) \to (0, \infty)$ where $\ell_\gamma([\rho])$ is the (real) translation distance of $\rho(\Gamma)$. We say that a closed curve intersects a bending lamination β_ν transversely if its geodesic representative in $\partial C_\nu(\rho)$ intersects β_ν transversely.

Theorem 1.3. *Suppose that $[\rho] \in \mathrm{QF}(S)$ is not fuchsian, $\Omega_\nu(\rho)$ is moderately bent (for some $\nu \in \{\pm\}$) and $w = w_\nu(\rho)$ is the infinitesimal bending deformation for $-i\beta_\nu$. If $\gamma \in \pi_1(S)$, then $d\ell_\gamma(w) \leq 0$ and $d\ell_\gamma(w) < 0$ if γ intersects β_ν transversely.*

Theorem 1.1 then follows immediately from Theorem 1.3 and a result of Sambarino [59, Lemma 2.32], see Section 4.

In order to prove Theorem 1.2 we need the following strengthening of Theorem 1.3.

Theorem 1.4. *Suppose that $[\rho] \in \mathrm{QF}(S)$ is not fuchsian and that $w = w_+(\rho) + w_-(\rho)$. If both $\Omega_+(\rho)$ and $\Omega_-(\rho)$ are moderately bent, then there exists $K > 0$ such that*

$$\mathrm{d}\ell_\gamma(w) \leq -K\ell_\gamma(\rho)$$

for all $\gamma \in \pi_1(S)$.

We now outline how Theorem 1.4 is used to prove Theorem 1.2. If $\rho : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ is quasifuchsian, then $\mathsf{u} : \pi_1(S) \to \mathfrak{sl}(2,\mathbb{C})$ is a cocycle for $\mathrm{Ad}\,\rho$ if

$$\mathsf{u}(\gamma\eta) = \mathsf{u}(\gamma) + \mathrm{Ad}\,\rho(\gamma)\mathsf{u}(\eta) \quad \text{for all} \quad \gamma, \eta \in \pi_1(S).$$

Cocycles for $\mathrm{Ad}\,\rho$ are in one-to-one correspondence with Affine representations with linear part $\mathrm{Ad}\,\rho$. Specifically, for $\mathsf{u} : \pi_1(S) \to \mathfrak{sl}(2,\mathbb{C})$, the map $F_{\rho,\mathsf{u}} : \pi_1(S) \to \mathrm{Aff}(\mathfrak{sl}(2,\mathbb{C}))$ given by

$$F_{(\rho,\mathsf{u})}(\gamma) = (\mathrm{Ad}\,\rho(\gamma), \mathsf{u}(\gamma)) \quad \text{for all} \quad \gamma \in \pi_1(S)$$

is an affine representation if and only if u is a cocycle for $\mathrm{Ad}\,\rho$.

Suppose that $[\rho] \in QF(S)$ and $v \in \mathsf{T}_{[\rho]}QF(S)$, then we choose a vector $\tilde{v} \in \mathsf{T}_\rho \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$ which projects to $v \in \mathsf{T}_{[\rho]}QF(S)$. If $\gamma \in \pi_1(S)$, then the function $A_\gamma : \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C})) \to \mathsf{PSL}(2,\mathbb{C})$ given by $A_\gamma(\rho) = \rho(\gamma)$ is a complex analytic map. We define

$$\mathsf{u}_{\tilde{v}} : \pi_1(S) \to \mathfrak{sl}(2,\mathbb{C}) \quad \text{by letting} \quad \mathsf{u}_{\tilde{v}}(\gamma) = dA_\gamma(\tilde{v})A_\gamma(\tilde{\rho})^{-1}$$

for all $\gamma \in \pi_1(S)$, which is a cocycle for $\mathrm{Ad}\,\rho$. Cocycles for $\mathrm{Ad}\,\rho$ are in one-to-one correspondence with tangent vectors in $\mathsf{T}_\rho \mathrm{Hom}(\pi_1(S), \mathsf{PSL}(2,\mathbb{C}))$. Different choices of representatives of $[\rho]$ and/or $\tilde{v}$ will produce representations conjugate to $F_{(\rho,\mathsf{u}_{\tilde{v}})}$.

Kassel and Smilga [42], see also Ghosh [32], developed a properness criterion for affine actions which is expressed in terms of the Margulis invariant spectrum, see Proposition 6.6. Theorem 1.4 will allow us to control the Margulis invariant spectrum of (ρ, u_w) when ρ and w satisfy the assumptions of the theorem. See Section 6 for details.

In order to obtain the explicit neighborhoods of the Fuchsian locus that we promised earlier we recall a numerical invariant which we can use to guarantee that $\Omega_\pm$ is moderately bent. If β_ν is a bending lamination and $L > 0$, we let $\|\beta_\nu\|_L$ be the supremum of the measure of any half-open arc of length less than L. The invariant $\|\beta_\nu\|_L$ is sometimes called the L-roundness of β and was originally introduced by Epstein, Marden and Markovic [28].

We use the techniques of Bridgeman-Canary-Yarmola [15] to show that there exist bounds on the L-roundness of β_ν which guarantee that Ω_ν is moderately bent. We define a threshold function $r : \mathbb{R}_+ \to \mathbb{R}_+$ such that if $\|\beta_\nu\|_L < r(L)$ the region Ω_ν is moderately bent. On $(0,1]$, r is the inverse of the function $y = x\sec(x)$ and for $x > 1$, $r(x) = x\mathrm{sech}(x)$. A simple analysis of the function $x\mathrm{sech}(x)$ shows that $r(x) \geq x\mathrm{sech}(x)$ for all x.

Theorem 1.5. *Let $\rho \in \mathrm{QF}(S)$ have bending laminations β_+ and β_-. If*

$$\|\beta_\nu\|_L < r(L) \quad \textit{for some} \quad L > 0,$$

then $\Omega_\nu(\rho)$ is moderately bent.

Our promised neighborhoods now have the following concrete form.

Corollary 1.6. *Let*

$$U(S) = \{[\rho] \in \mathrm{QF}(S) : \|\beta_+\|_{L_1} < r(L_1) \ \textit{ for some } \ L_1 > 0 \ \textit{ or } \ \|\beta_-\|_{L_2} < r(L_2) \ \textit{ for some } \ L_2 > 0\}.$$

and

$$V(S) = \{[\rho] \in \mathrm{QF}(S) : \|\beta_+\|_{L_1} < r(L_1) \ \textit{ for some } \ L_1 > 0 \ \textit{ and } \ \|\beta_-\|_{L_2} < r(L_2) \ \textit{ for some } \ L_2 > 0\}.$$

(1) *$U(S)$ is an open neighborhood of the Fuchsian locus, so that if $[\rho] \in U(S)$ is not fuchsian, then $[\rho]$ is not a critical point of the entropy function h.*
(2) *$V(S)$ is an open neighborhood of the Fuchsian locus, so that if $[\rho] \in V(S)$ is not fuchsian, then $\mathrm{Ad}\,\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$.*

We now discuss the explicit constants we obtain. For example,

$$r(1) \approx .739$$

For comparison, it follows from [15, Thm. 4.1] that for any $\rho \in \mathrm{QF}(S)$ one has

$$\|\beta_\pm\|_L \le 2\cos^{-1}\left(-\sinh(L/2)\right).$$

In particular,

$$\|\beta_\pm\|_1 \le 2\cos^{-1}(-\sinh(1/2)) \approx 4.2379.$$

To consider how close $r(L)$ is to being maximal, we describe the following elementary example. Take a horocycle H in a hyperbolic plane $P \subseteq \mathbb{H}^3$ and let $\gamma : \mathbb{R} \to P$ be a piecewise geodesic with vertices on H and each edge of length L. We then let C be the convex pleated plane containing γ and perpendicular to P. Then a simple calculation gives that C has bending lamination β with

$$\|\beta\|_L = 2\sin^{-1}\left(\tanh(L/2)\right).$$

It follows that there are non-embedded convex pleated planes with $\|\beta\|_1$ arbitrarily close to $2\sin^{-1}\left(\tanh(1/2)\right) \simeq .9607$.

In Section 9 we discuss other bounds guaranteeing that our domains of discontinuity are moderately bent. These bounds are in terms of the Schwarzian derivative, the Teichmüller distance between the two boundary components and the quasiconformal distortion of $\Lambda(\rho)$.

One may also obtain the following applications to more general complex Lie groups. If G is a complex simple Lie group and Γ is a finitely generated group, let $\mathcal{C}(\Gamma, \mathsf{G})$ be the space of pairs (ρ, u) where $\rho : \Gamma \to \mathsf{G}$ is a representation and u is a cocycle for $\mathrm{Ad}\,\rho$. We give $\mathcal{C}(\Gamma, \mathsf{G})$ the compact-open topology. If ρ is a smooth point of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$, then the space of cocycles for $\mathrm{Ad}\,\rho$ is canonically identified with $\mathsf{T}_\rho\mathrm{Hom}(\pi_1(S), \mathsf{G})$. In general, the space of cocycles for $\mathrm{Ad}\,\rho$ is identified with Zariski tangent space of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ at ρ (see [36]).

Corollary 1.7. *Let G be a complex simple Lie group with Lie algebra $\mathfrak{g}$.*

i) *The subset of $\mathcal{C}(\pi_1(S), \mathsf{G})$ consisting of pairs (ρ, u) such that $F_{(\rho,\mathsf{u})}(\pi_1(S))$ acts properly discontinuously on $\mathfrak{g}$ has non-empty interior.*
ii) *The set of representations of $\pi_1(S)$ into $\mathsf{G} \times \mathsf{G}$ that are discrete and faithful in each factor, and whose action on G via left/right multiplication is proper, has non-empty interior. Moreover, its closure contains product representations $\rho \times \rho$ where ρ is totally Anosov.*

The key tool in the proof of (i) is the fact that the normalized Margulis spectrum varies continuously (see Sambarino [59, Lemma 2.34]). The proof of (ii) makes crucial use of a properness criterion of Benoist [3] and Kobayashi [44]. Danciger, Guéritaud and Kassel [22, Prop. 1.8] previously established part (ii) of Corollary 1.7 in the case when $\mathsf{G} = \mathsf{PSL}(2, \mathbb{C})$.

Historical background: Auslander [2] conjectured that if Γ acts properly discontinuously and cocompactly on $\mathbb{R}^n$ by affine transformations, then Γ is virtually solvable. The conjecture remains open, but has been established when $n = 3$ by Fried-Goldman [30], when $n = 4$ and $n = 5$ by Tomanov [64] and when $n = 6$ by Abels, Margulis, and Soifer [1]. Milnor [54] asked whether, in analogy with the Bieberbach theorems, something similar might be true for actions which are not cocompact. Margulis [51, 52] produced the first examples of proper affine actions by non-abelian free groups on $\mathbb{R}^3$, which are now called Margulis space-times. Drumm [24], Charette-Drumm-Goldman [18], and Drumm-Goldman [25] introduced a geometric viewpoint on Margulis' construction and produced large classes of new examples. Goldman, Labourie and Margulis [38] gave an exact criterion for when affine actions of free groups on $\mathbb{R}^3$ are proper (see Ghosh-Treib [34] for extensions to $\mathbb{R}^{2n+1}$). Danciger, Guéritaud and Kassel [20, 21] gave a complete classification of Margulis space-times with convex cocompact linear part and showed that their quotients are all homeomorphic to the interior of a handlebody.

Smilga [61] extended Goldman, Labourie and Margulis' properness criterion to any real semisimple Lie group G of the non-compact type. He constructed a proper affine action of a non-abelian free group on the Lie algebra $\mathfrak{g}$ whose linear part is Zariski dense in $\mathrm{Ad}(\mathsf{G})$. Burelle and Zager Korenjak [16, 66] further analyzed higher-dimensional Margulis space times.

It is natural to ask which other groups admit proper affine actions. Mess [53] showed that a closed surface group cannot have a proper affine action on $\mathbb{R}^3$ (see [53]). Danciger and Zhang [23] showed that a proper affine action of a surface group on $\mathbb{R}^d$ cannot have a linear part which is a Hitchin representation into $\mathsf{SL}(d,\mathbb{R})$. (see also Labourie [49]). In a major breakthrough, Danciger, Guéritaud and Kassel [22] proved that any right-angled Coxeter group on k generators admits a proper affine actions on $\mathbb{R}^{k(k-1)/2}$. They also show that every hyperbolic surface group admits a proper affine action on $\mathbb{R}^6$.

We briefly describe the construction of Danciger, Guéritaud and Kassel in [22] for producing proper affine actions of hyperbolic surface groups on $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{R}^6$ whose linear part is the adjoint action of the a quasifuchsian group. A surface group of genus g is an index 4 subgroup of the fuchsian Coxeter group which is the reflection group of a $(2g+2)$-gon all of whose internal angles are $\frac{\pi}{2}$. They describe an explicit path in the deformation space of the Coxeter group within $\mathsf{PSL}(2,\mathbb{C})$ and show that the nonfuchsian points on this path give rise to proper affine actions.

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2. Background

2.1. **Geodesic currents and measured laminations.** Bonahon [9] introduced the space $\mathcal{C}(X)$ of geodesic currents on a closed hyperbolic surface X as a natural closure of the space of weighted collections of closed geodesics on X. Let $G(\mathbb{H}^2)$ be the space of unoriented geodesics in $\mathbb{H}^2$. If $X = \mathbb{H}^2/\Gamma$ is a hyperbolic surface, a *geodesic current* on X is a locally finite Γ-invariant measure on $G(\mathbb{H}^2)$. We endow the space $\mathcal{C}(X)$ of geodesic currents on X with the weak* topology. Any closed geodesic α (with weight one) on X gives rise to a geodesic current by considering the measure which has an atom of weight one on each geodesic in $\mathbb{H}^2$ which covers α.

Bonahon [9] showed that one may continuously extend the length function on geodesics to a length function

$$L_X : \mathcal{C}(X) \to [0,\infty).$$

We give a brief description of his construction. Let $\mathsf{PT}(\mathbb{H}^2)$ denote the projective tangent bundle of $\mathbb{H}^2$ and consider the trivial fiber bundle $\pi : \mathsf{PT}(\mathbb{H}^2) \to G(\mathbb{H}^2)$ mapping a project tangent vector a point to the associated unoriented geodesic through the point. The fiber above a geodesic g can be identified with g and given its length measure ℓ. So, if $\mu \in \mathcal{C}(X)$, then we obtain a Γ-invariant measure on $\mathsf{PT}(\mathbb{H}^2)$ which is locally $\mu \times \ell$. This measure descends to a measure $\widehat{\mu \times \ell}$ on $\mathsf{PT}(X)$. Bonahon defines $L_X(\mu)$ to be the total mass of $\widehat{\mu \times \ell}$.

Given homotopy classes α and β of closed geodesics on X, one defines their *geometric intersection number* $i(\alpha,\beta)$ to be the number of transverse intersection points of α and β. Bonahon also showed that one may continuously extend the intersection number to a continuous function on $\mathcal{C}(X) \times \mathcal{C}(X)$. If $\mu, \nu \in \mathcal{C}(X)$, then then $\mu \times \nu$ gives a measure on $G(\mathbb{H}^2) \times G(\mathbb{H}^2)$ invariant under the diagonal action of Γ. Letting

$$D(\mathbb{H}^2) = \{(\vec{u},\vec{v}) \in \mathsf{PT}_x(\mathbb{H}^2) \times \mathsf{PT}_x(\mathbb{H}^2) \mid x \in \mathbb{H}^2 \text{ and } \vec{u} \neq \vec{v}\}$$

we can identify $D(\mathbb{H}^2)$ with a subset of $G(\mathbb{H}^2)\times G(\mathbb{H}^2)$ by mapping $(\vec{u},\vec{v})$ to the pair of geodesics tangent to them. Then

$$D(X) = D(\mathbb{H}^2)/\Gamma = \{(\vec{u},\vec{v}) \in \mathsf{PT}_x(X)\times \mathsf{PT}_x(X) \mid x\in X \text{ and } \vec{u}\neq\vec{v}\}.$$

Since the pullback of the measure $\mu\times\nu$ to $D(\mathbb{H}^2)$ is invariant under the action of Γ it projects to a measure $\widehat{\mu\times\nu}$ on $D(X)$. Bonahon defines $i_X(\mu,\nu)$ to be the total mass of $\widehat{\mu\times\nu}$ on $D(X)$.

If S is a closed topological surface of genus at least 2, we may define the space $\mathcal{C}(S)$ as the space of $\pi_1(S)$-invariant locally finite measures on the space of unordered pairs of distinct points in the Gromov boundary $\partial_\infty\pi_1(S)$ of $\pi_1(S)$. Given $(X,h)\in\mathcal{T}(S)$, there is a well-defined h_*-equivariant homeomorphism $\xi:\partial_\infty:\pi_1(S)\to\partial\mathbb{H}^2$, which gives rise to a canonical identification of $\mathcal{C}(S)$ to $\mathcal{C}(X)$. We notice that L_X does not induce a well-defined function on $\mathcal{C}(S)$, since the length function depends on the underlying hyperbolic structure. However, the definition of intersection number does not depend on the choice of hyperbolic structure so we have a well-defined continuous function

$$i:\mathcal{C}(S)\times\mathcal{C}(S)\to[0,\infty)$$

where for any $(X,h)\in\mathcal{T}(S)$, we take $i(\mu,\nu)=i_X(\xi_*(\mu),\xi_*(\nu))$.

Notice that a closed geodesic α on X is simple if and only if $i(\alpha,\alpha)=0$. More generally, we may consider the space $\mathcal{ML}(X)$ of measured geodesic laminations on X to be the set of geodesic currents of self-intersection zero. If $\mu\in\mathcal{ML}(X)$ then its support $\mathrm{supp}(\mu)$ is a closed set foliated by disjoint complete geodesics, i.e. a geodesic lamination. Any interval α transverse to the support, inherits a measure by taking the μ-measure of all the geodesics transverse to α. This gives the *transverse measure* associated to μ. Since intersection number is independent of the hyperbolic structure, $\mathcal{ML}(S)$ is also well-defined.

2.2. **The convex core and bending laminations.** If ρ is quasifuchsian, but not fuchsian, there exists an orientation-preserving homeomorphism

$$j_\rho : S\times[0,1]\to C(\rho)$$

in the homotopy class of ρ. Thurston [63] showed that the intrinsic metric on each component of $\partial C(\rho)$ is hyperbolic. Moreover, there exist measured laminations β_+ and β_- on $\partial C_+(\rho)=j_\rho(S\times\{1\})$ and $\partial C_-(\rho)=j_\rho(S\times\{0\})$ so that $\partial C_\pm(\rho)$ is totally geodesic on the complement of the support $\mathrm{supp}(\beta_\pm)$ of $\beta_\pm$ and each leaf of $\mathrm{supp}(\beta_\pm)$ is mapped to a geodesic. (See Epstein-Marden [27] for complete proofs.)

We now briefly describe the transverse bending measures given by β_ν for $\nu\in\{\pm\}$. For a complete description see Epstein-Marden [27]. It will be easiest to work in the universal cover. Let $\tilde{\partial}C_\nu(\rho)$ denote the pre-image of $\partial C_\nu(\rho)$ in $\partial CH(\rho)$. A *support half-space* to $\tilde{\partial}C_\nu(\rho)$ at $x\in\partial C_\nu(\rho)$ is a half-space H such that the interior of H is disjoint from the convex hull $CH(\rho)$ and $x\in\partial H$. The boundary ∂H of a support half-plane, is called a *support plane*. If $\alpha:[0,1]\to\tilde{\partial}C_\nu(\rho)$ is an arc transverse to β_ν (so that $\alpha(0),\alpha(1)\notin\mathrm{supp}(\beta_\nu)$), $\mathcal{P}=\{0=t_0<t_1<\ldots<t_n=1\}$ is a partition of $[0,1]$ and $\mathcal{H}=\{H_{t_i}\}$ is a collection of support planes to $\tilde{\partial}C_\nu(\rho)$ at $\alpha(t_i)$, we define

$$i_{\mathcal{P},\mathcal{H}}(\alpha,\beta_\nu)=\sum_{i=0}^{n-1}\theta_i$$

where θ_i is the angle between H_{t_i} and $H_{t_{i+1}}$. For the definition we need to restrict to pairs $(\mathcal{P},\mathcal{H})$ with the property that H_{t_i} intersects $H_{t_{i+1}}$. Furthermore it is natural to restrict to pairs $(\mathcal{P},\mathcal{H})$ such that if $t\in[t_i,t_{i+1}]$ and H_t is a support half-space to $\alpha(t)$ then H_t intersects both H_{t_i} and $H_{t_{i+1}}$.

We define the bending measure, as the limit over any sequence $\{(\mathcal{P}_n,\mathcal{H}_n)\}$ so that the mesh size of P_n tends to zero. Since the intersection number $i_{\mathcal{P},\mathcal{H}}(\alpha,\beta_\nu)$ is monotonically decreasing under refining the partition, we can also define the bending measure for α to be

$$i(\alpha,\beta_\nu)=\inf_{\mathcal{P},\mathcal{H}} i_{\mathcal{P},\mathcal{H}}(\alpha,\beta_\nu).$$

2.3. **Complex length.** If g and h are oriented geodesics in $\mathbb{H}^3$ which do not share an endpoint, then one can define a (unsigned) *complex distance* $\sigma(g,h)$ between them. The real part of $\sigma(g,h)$ is simply the distance between g and h, the complex part is the angle between g and the parallel translate of h along the unique common perpendicular joining g to h measured counterclockwise in the plane spanned by g and the parallel translate of h. The unsigned complex distance naturally lies in $\mathbb{C}/2\pi i\mathbb{Z}$ and varies smoothly over the space of pairs of oriented geodesics.

Kouroniotis [46, Lemma 2.1] shows that the unsigned complex distance is determined by the relation

$$\cosh\sigma(g,h) = \frac{[g_-,h_-,h_+,g_+]+1}{[g_-,h_-,h_+,g_+]-1}$$

where $[\cdot,\cdot,\cdot,\cdot]$ is the cross ratio on the Riemann sphere given by

$$[u,p,q,v] = \frac{(u-q)(v-p)}{(u-p)(v-q)}.$$

It follows, in particular, that $\cosh\sigma$ is a well-defined smooth complex valued function. As this is the only function that appears in our calculations, we don't need to consider evaluating σ directly. See Series [60] for a helpful discussion of unsigned complex distance.

Notice that if $(h_-,h_+)=(0,\infty)$, and $(g_-,g_+)=(u,v)$, then

$$[g_-,h_-,h_+,g_+] = \frac{v}{u} \quad \text{and} \quad \cosh\sigma(g,h) = \frac{v+u}{v-u}.$$

The following formula for the imaginary part of $\cosh\sigma(g,h)$ will be used later:

$$\Im\Big(\cosh\sigma(g,h)\Big) = -\frac{2\Im[g_-,h_-,h_+,g_+]}{|[g_-,h_-,h_+,g_+]-1|^2}. \tag{1}$$

One can define the *complex length* $\mathcal{L}(\gamma)$ of a hyperbolic isometry γ of $\mathbb{H}^3$. The real part of $\mathcal{L}(\gamma)$ is the translation length of γ along its axis, while its imaginary part is its rotational angle. So, the complex length $\mathcal{L}(\gamma)$ is only defined modulo $2\pi i\mathbb{Z}$.

If $z\in\mathbb{C}$ and $A(z)$ is the hyperbolic transformation $w\to e^z w$, then $\mathcal{L}(A(z))=z$. If g is an oriented geodesic in $\mathbb{H}^3$ and $t\in\mathbb{C}$, we let Q be an isometry mapping the pair $(0,\infty)$ to the pair (g_-,g_+) and set $R(g,z)=QA(z)Q^{-1}$. Notice that $R(g,z)$ is well-defined, varies holomorphically in z and $\mathcal{L}(R(g,z))=z$.

2.4. **Bending deformations.** Kourouniotis [47] studied the behavior of bending deformation using the theory of complex measured laminations. If X is a hyperbolic structure on a closed surface, a *complex measured lamination* μ on X is a complex-valued, locally finite measure on $G(\mathbb{H}^2)$ whose support is a geodesic lamination. The space $\mathcal{ML}_{\mathbb{C}}(X)$ of complex measured lamination is topologized with the weak* topology on measures. A complex measure lamination μ induces a complex measure on arcs transverse to $\mathrm{supp}(\mu)$.

We will be interested in the subspace $\mathcal{ML}^{++}(X)$ of $\mathcal{ML}_{\mathbb{C}}(X)$ consisting of measures which take values with non-negative real and imaginary parts. Notice also that $\mathcal{ML}^{++}(S)$ is well-defined.

We first define the bending deformation of a quasifuchsian representation ρ along $\mu\in\mathcal{ML}^{++}(S)$ when the support of μ has finitely many leaves. Since X is a hyperbolic structure on S, $X=\mathbb{H}^2/\Gamma$ and this gives an identification of $\partial_\infty\pi_1(S)$ with $\partial\mathbb{H}^2$. Consider the ρ-equivariant map $\xi_\rho:\partial\mathbb{H}^2\to\partial\mathbb{H}^3$. The map ξ induces a ρ-equivariant map $\xi_\rho^*:\mathcal{G}(\mathbb{H}^2)\to\mathcal{G}(\mathbb{H}^3)$ where $\mathcal{G}(\mathbb{H}^n)$ is the space of oriented geodesics in $\mathbb{H}^n$. We fix a basepoint $x_0\in X$, in the complement of $\mathrm{supp}(\mu)$ and a preimage $\tilde{x}_0$ of x_0 in $\mathbb{H}^2$. Given $\gamma\in\pi_1(X,x_0)$, let $m_1,\ldots,m_n$ be the geodesics in the support of $\tilde{\mu}$ intersecting $[\tilde{x}_0,\gamma(\tilde{x}_0)]$ with atomic measures $a_1,\ldots a_m$. We order the leaves from right to left and orient each m_i so that they cross $[\tilde{x}_0,\gamma(\tilde{x}_0)]$ from left to right. We then define, for all $z\in\mathbb{C}$,

$$\rho_{z\mu}(\gamma) = R(\xi_*(m_1),za_1)R(\xi_*(m_2),za_2)\ldots R(\xi_*(m_{n-1},za_{m-1})R(\xi_*(m_n),za_n)\rho(\gamma).$$

Kouroniotis observes that $\rho_{z\mu}$ is a representation for all z and is quasifuchsian for z lying in an open neighborhood of 0 in $\mathbb{C}$.

Kouroniotis showed that if $\{\mu_n\}$ is a sequence of finite leaved complex measured laminations converging to $\mu \in \mathcal{ML}_{\mathbb{C}}(X)$, then $B_{z\mu_n}$ converges to $B_{z\mu}$ and we obtain a map

$$B_\mu : \mathrm{QF}(S) \times \mathbb{C} \to \mathfrak{X}(S) \quad \text{given by} \quad B_\mu([\rho], z) = [\rho_{z\mu}]$$

which is holomorphic in z (see [47]). The associated holomorphic bending vector field $T_\mu : \mathrm{QF}(S) \to T(\mathrm{QF}(S))$ is defined to be

$$T_\mu([\rho]) = \left.\frac{\partial}{\partial z}\right|_{z=0} B_\mu([\rho], z)$$

Kourouniotis further proved

Theorem 2.1. (Kourouniotis [47, Theorem 3]) *The map $T : \mathcal{ML}^{++}(S) \times \mathrm{QF}(S) \to T(\mathrm{QF}(S))$ given by $T(\mu, [\rho]) = T_\mu([\rho])$ is continuous and holomorphic in $[\rho]$.*

3. Variation of complex length

In this section, we obtain our formula for the variation of the complex length function during a bending defonation along a measured lamination μ. Kouroniotis [46] already obtained a formula when the support of μ has finitely many leaves.

Theorem 3.1. (Kouroniotis [46, Thm 4.1]) *If ρ is quasifuchsian, $\mu \in \mathcal{ML}(S)$ has finite-leaved support, $S \cong X = \mathbb{H}^2/\Gamma$, $\gamma \in \pi_1(X, x_0)$, $x_0 \notin \mathrm{supp}(\mu)$ and $\mathcal{L}_\gamma(z) = \mathcal{L}(\rho_{z\mu}(\gamma))$, then*

$$\mathcal{L}'_\gamma(0) = \sum_{j=1}^n a_j \cosh(\sigma((\xi_\rho)_*(a(\gamma)), (\xi_\rho)_*(m_j))$$

where $[\tilde{x}_0, \gamma(\tilde{x}_0)]$ intersects $\tilde{\mu}$ in the leaves $\{m_1, \ldots, m_n\}$ (ordered from right to left and oriented so they cross $[\tilde{x}_0, \gamma(\tilde{x}_0)]$ from left to right) with atomic weights $\{a_1, \ldots, a_n\}$ and $a(\gamma)$ is the axis of γ.

We will use Theorem 2.1 to extend Kourouniotis' formula to the general case. We first re-interpret Kourouniotis' formula in a more measure-theoretic fashion. If $\gamma \in \pi_1(X, x_0)$ is non-trivial, we let

$$D_\gamma(X) = \{(\vec{u}, \vec{v}) \in D(X) : x \in \gamma^*, \ \vec{u} \text{ is tangent to } \gamma^* \text{ at } x, \text{ and } \vec{u} \neq \vec{v}\}.$$

If $(\vec{u}, \vec{v}) \in D_\gamma(X)$, we lift $\vec{u}$ and $\vec{v}$ to a pair $(\vec{U}, \vec{V}) \in \mathsf{PT}_{\tilde{x}}(\mathbb{H}^2) \times \mathsf{PT}_{\tilde{x}}(\mathbb{H}^2)$ with $\vec{U}$ tangent to the axis $a_U(\gamma)$ of γ at a lift $\tilde{x}$ of x. Then let g_V be an oriented geodesic in $\mathbb{H}^2$ which has $\vec{V}$ as a tangent vector and crosses $a_U(\gamma)$ from left to right. We define a continuous function

$$\sigma_{\rho,\gamma} : D_\gamma(X) \to \mathbb{C}/2\pi i\mathbb{Z}$$

given by

$$\sigma_{\rho,\gamma}(\vec{u}, \vec{v}) = \sigma\Big((\xi_\rho)_*(a_U(\gamma)), (\xi_\rho)_*(g_V)\Big).$$

If we regard γ as a measured geodesic lamination with atomic measure, then the support of the measure $\widehat{\gamma \times \mu}$ on $D(X)$ lies within $D_\gamma(X)$, so we may view $\widehat{\gamma \times \mu}$ as a measure on $D_\gamma(X)$. Then notice that

$$\mathcal{L}'_\gamma(0) = \int_{D_\gamma(X)} \cosh \sigma_{\rho,\gamma} \ d(\widehat{\gamma \times \mu}).$$

Notice that this formula does not depend on the choice of X.

We use Kourouniotis' later work [47] on the continuity of bending to extend this formula to the general case.

Theorem 3.2. *If ρ is quasifuchsian, $\mu \in \mathcal{ML}^{++}(S)$, $S \cong X = \mathbb{H}^2/\Gamma$, $\gamma \in \pi_1(X, x_0)$ and $\mathcal{L}_\gamma(z) = \mathcal{L}(\rho_{z\mu}(\gamma))$, then*

$$\mathcal{L}'_\gamma(0) = \int_{D_\gamma(X)} \cosh \sigma_{\rho,\gamma} \ d(\widehat{\gamma \times \mu}).$$

Proof. First notice that our formula holds trivially when $i(\gamma,\mu)=0$, since both sides are clearly equal to 0. So we will assume that $i(\gamma,\mu)\neq 0$.

Fix $(X,h)\in\mathcal{T}(S)$ and let $\{\mu_n\}$ be a sequence of finite-leaved complex measured laminations in $\mathcal{ML}^{++}(X)$ which converge to μ. We claim that there exists $\theta>0$ and N so that if $n\geq N$ and $(\vec{u},\vec{v})$ is in the support of $\widehat{\gamma\times\mu_n}$, then the (un-oriented) angle $\angle\vec{u},\vec{v}$ between $\vec{u}$ and $\vec{v}$ lies between θ and $\pi-\theta$. If not, then γ lies in the Hausdorff limit of a subsequence of $\{\mathrm{supp}(\mu_n)\}$. Since the support of μ is contained in the Hausdorff limit of any subsequence of $\{\mathrm{supp}(\mu_n)\}$, this would imply that $i(\gamma,\mu)=0$.

Let $\mathcal{L}^n_\gamma(z)=\mathcal{L}(\rho_{z\mu_n}(\gamma))$ and $\mathcal{L}_\gamma(z)=\mathcal{L}(\rho_{z\mu}(\gamma))$. Each $\mathcal{L}^n_\gamma$ is holomorphic in z (by Theorem 2.1) and $\{\mathcal{L}^n_\gamma\}$ converges uniformly on compact sets to $\mathcal{L}_\gamma$. Thus,

$$\mathcal{L}'_\gamma(0)=\lim(\mathcal{L}^n_\gamma)'(0).$$

Moreover, Theorem 3.1 implies that

$$(\mathcal{L}^n_\gamma)'(0)=\int_{D_\gamma(X)}\cosh\sigma_{\sigma,\rho}\ d(\widehat{\gamma\times\mu_n})$$

for all n. Let

$$D_{\gamma,\theta}(X)=\{(\vec{u},\vec{v})\in D_\gamma(X):\pi-\theta\geq\angle\vec{u},\vec{v}\geq\theta\}.$$

For all $n\geq N$,

$$(\mathcal{L}^n_\gamma)'(0)=\int_{D_{\gamma,\theta}(X)}\cosh\sigma_{\sigma,\rho}\ d(\widehat{\gamma\times\mu_n})$$

Since, $D_{\gamma,\theta}(X)$ is compact and $\{\gamma\times\mu_n\}$ converges to $\gamma\times\mu$ in the weak*-topology, we see that

$$\lim_{n\to\infty}(\mathcal{L}^n_\gamma)'(0)=\int_{D_{\gamma,\theta}(X)}\cosh\sigma_{\sigma,\rho}\ d(\widehat{\gamma\times\mu})=\int_{D_\gamma(X)}\cosh\sigma_{\sigma,\rho}\ d(\widehat{\gamma\times\mu})$$

which completes the proof. □

4. Critical points of the entropy function

In this section we show that if one of the components of the domain of discontinuity of a nonfuchsian quasifuchsian group is moderately bent, then it is not a critical point of the entropy function

Work of Sambarino [59] provides a criterion guaranteeing that ρ is not a critical point. If we consider $[\rho]\in\mathrm{QF}(S)$ and a non-zero variation $\vec{v}\in T_{[\rho]}\mathrm{QF}(S)$, then the link between variation of entropy and variation of lengths is given by the *set of normalized variations*

$$\mathbb{V}_{\vec{v}}=\overline{\left\{\frac{\mathrm{d}\ell_\gamma(\vec{v})}{\ell_\gamma}:\gamma\in\pi_1(S)\right\}}\subset\mathbb{R}.$$

This is a closed interval and if ρ is not fuchsian it has non-empty interior. Moreover, in this case $-\mathrm{d}h(\vec{v})/h(\rho)\in\mathrm{int}(\mathbb{V}_{\vec{v}})$ (see [59, Lemma 2.32]). The following criterion follows immediately.

Proposition 4.1. (Sambarino [59]) *Suppose that $[\rho]\in\mathrm{QF}(S)$ is not fuchsian and $\vec{v}\in T_{[\rho]}\mathrm{QF}(S)$ is not zero. If $\mathrm{d}\ell_\gamma(\vec{v})\leq 0$ for all $\gamma\in\pi_1(S)$, then $\mathrm{d}h(\vec{v})\neq 0$, in particular h is not critical at $[\rho]$.*

We can use our formula for the derivative of complex length and our angle bounds to produce such vectors.

Theorem 1.3. *Suppose that $[\rho]\in QF(S)$ is not fuchsian, $\Omega_\nu(\rho)$ is moderately bent (for some $\nu\in\{\pm\}$) and $w=w_\nu(\rho)$ is the infinitesimal bending deformation for $-i\beta_\nu$. If $\gamma\in\pi_1(S)$, then $d\ell_\gamma(w)\leq 0$ and $d\ell_\gamma(w)<0$ if γ intersects β_ν essentially.*

Notice that Theorem 1.1 follows immediately from Theorem 1.3 and Proposition 4.1.

Proof. By Theorem 3.2,

$$d\ell_\gamma(w)=\Re(d\mathcal{L}_\gamma(w))=\Re\left(\int_{D_\gamma(X)}\cosh\sigma_{\rho,\gamma}\ d(\widehat{\gamma\times -i\beta_\nu})\right)=\int_{D_\gamma(X)}\Im\Big(\cosh\sigma_{\rho,\gamma}\Big)\ d(\widehat{\gamma\times\beta_\nu}).$$

In particular, $d\ell_\gamma(w) = 0$ if γ does not intersect β_ν essentially. Recall that given $(\vec{u}, \vec{v}) \in D_\gamma(X)$, we lift $(\vec{u}, \vec{v})$ to a pair of projective tangent vectors $(\vec{U}, \vec{V}) \in \mathsf{PT}\mathbb{H}^2$ which are tangent to geodesics $a_U(\gamma)$ and g_V. Then

$$\sigma_{\rho,\gamma}(\vec{u}, \vec{v}) \ = \sigma((\xi_\rho)_*(a_U(\gamma)), (\xi_\rho)_*(g_V)).$$

Denoting the geodesics as an oriented pair of endpoints, we let $(u_-, u_+) = (\xi_\rho)_*(a_U(\gamma))$ and $(v_-, v_+) = (\xi_\rho)_*(g_V)$. By Equation (1)

$$\Im\Big(\cosh \sigma_{\rho,\gamma}\Big)(\vec{u}, \vec{v}) \ = \frac{-2\Im\Big([u_-, v_-, v_+, u_+]\Big)}{|[u_-, v_-, v_+, u_+] - 1|^2}.$$

Thus it suffices to show that $\Im\Big([u_-, v_-, v_+, u_+]\Big) > 0$ for all $(\vec{u}, \vec{v}) \in \operatorname{supp}(\widehat{\gamma \times \beta_\nu})$. Notice that, since g_V crosses $a_U(\gamma)$ from left to right, the vertices occur in the order u_-, v_-, u_+, v_+ on $\Lambda(\rho)$.

For a bending line with endpoints the bending pair (v_-, v_+), we can choose a support plane P which intersects $\Lambda(\rho)$ only at v_- and v_+. (If the bending line has a unique support plane it has this property, while if not we can take any support plane which is not left-most or right-most.) We may assume that $(v_-, v_+) = (0, \infty)$ that the plane P lying above the real axis $\mathbb{R}$ is a support plane for $CH(\Lambda(\rho))$, and the upper half-plane is the boundary disk of the support half-space bounded by P. It follows that Ω_ν contains the upper half-plane and that u_- and u_+ are in the lower half-plane. Since Ω_ν is moderately bent, there exists a line L through 0 and ∞ which intersects $\Lambda(\rho)$ transversely and only at 0 and ∞. With our conventions u_- lies to the left of L (i.e. the side containing the negative real axis) and u_+ lies to the right of L (see Figure 1). As the cross-ratio is invariant under rotation, we apply a clockwise rotation r about 0 so that $z_+ = r(u_+)$ lies on the positive real axis and let $z_- = r(u_-)$. The angle of rotation is less than the angle the portion of L below the real axis makes with the positive real axis, so $\Im(z_-) < 0$. So,

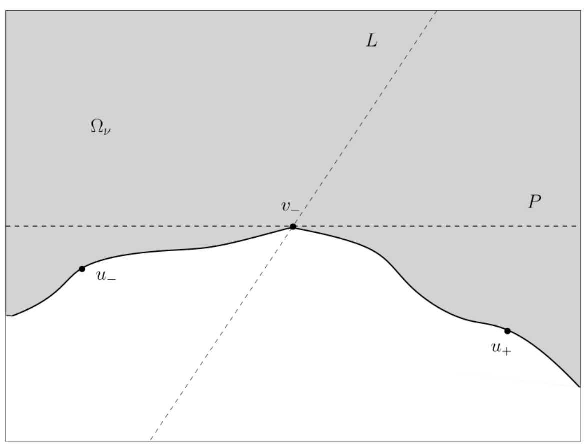

FIGURE 1. Domain Ω_ν

$$[u_-, v_-, v_+, u_+] = [z_-, 0, \infty, z_+] = \frac{z_+}{z_-} = \frac{z_+\overline{z_-}}{|z_-|^2}.$$

Moreover,

$$\Im\Big(z_+\overline{z_-}\Big) = -z_+\Im(z_-) > 0, \quad \text{so} \quad \Im\Big([u_-, v_-, v_+, u_+]\Big) > 0,$$

which completes the proof. □

5. Binding pairs of laminations and δ-intersection number

In this section we prove Theorem 1.4 which shows that lengths of curves decrease uniformly in the direction $w(\rho) = w_+(\rho) + w_-(\rho) \in \mathsf{T}_{[\rho]}QF(S)$.

We say that a pair (α, β) of measured laminations on S is *binding* if

$$i(\alpha,\mu) + i(\beta,\mu) > 0$$

whenever μ is a non-trivial geodesic current on S. If $(\vec{u},\vec{v}) \in D(X)$, let $\angle \vec{u},\vec{v}$ is the unoriented angle between $\vec{u}$ and $\vec{v}$. In particular, $(\vec{u},\vec{v}) \in (0,\pi))$. If $\delta \in \left(0,\frac{\pi}{2}\right)$, we define

$$D_\delta(X) = \{(\vec{u},\vec{v}) \in D(X) : \pi - \delta \geq \angle \vec{u},\vec{v} \geq \delta\}$$

If $\alpha, \mu \in \mathcal{C}(X)$ and $\delta > 0$, we define $i_\delta(\alpha,\mu)$ to be the total mass of the points in $D_\delta(X)$ with respect to the quotient measure of $\alpha \times \mu$. Notice that while intersection number is independent of the base hyperbolic structure, the δ-intersection number depends crucially on the underlying structure.

Lemma 5.1. *Let (α,β) be a binding pair of measured laminations on a closed hyperbolic surface X. There exists a$\delta > 0$ and $C > 0$ so that if $\mu \in \mathcal{C}(X)$, then*

$$i_\delta(\alpha,\mu) + i_\delta(\beta,\mu) \geq C\ell_X(\mu).$$

Proof. Suppose no such δ and C exist. Then, since ℓ_X and intersection number both scale linearly, there exist a sequence $\{\mu_n\}$ of unit length geodesic currents so that

$$i_{\frac{1}{n}}(\alpha,\mu_n) + i_{\frac{1}{n}}(\beta,\mu_n) \to 0.$$

Since the space $\mathcal{C}_1(X)$ of unit length geodesic currents is homeomorphic to the space of projective geodesic currents $\mathcal{PC}(S)$, $\mathcal{C}_1(X)$ is compact (see [9]). We may assume that $\mu_n \to \mu \in \mathcal{C}_1(X)$. Since (α,β) is binding,

$$i(\alpha,\mu) + i(\beta,\mu) > 0.$$

Since $D(X) = \bigcup_{n\in\mathbb{N}} D_{\frac{1}{n}}(X)$, we see that there exists $N \in \mathbb{N}$ so that

$$i_{\frac{1}{N}}(\alpha,\mu) + i_{\frac{1}{N}}(\beta,\mu) > 0.$$

By considering a continuous bump function supported on $D_{\frac{1}{2N}}(X)$ and equal to 1 on $D_{\frac{1}{N}}(X)$, we see that

$$\liminf_{n\to\infty}\left(i_{\frac{1}{2N}}(\alpha,\mu_n) + i_{\frac{1}{2N}}(\beta,\mu_n)\right) \geq i_{\frac{1}{N}}(\alpha,\mu) + i_{\frac{1}{N}}(\beta,\mu) > 0$$

which produces a contradiction as

$$\liminf_{n\to\infty}\left(i_{\frac{1}{2N}}(\alpha,\mu_n) + i_{\frac{1}{2N}}(\beta,\mu_n)\right) \leq \liminf_{n\to\infty}\left(i_{\frac{1}{n}}(\alpha,\mu_n) + i_{\frac{1}{n}}(\beta,\mu_n)\right) = 0.$$

□

We can now return to our bounds on derivatives and observe that they can be made uniform with respect to δ-intersection number.

Lemma 5.2. *Suppose that $[\rho] \in \mathrm{QF}(S)$ is not fuchsian, Ω_ν is moderately bent for some $\nu \in \{\pm\}$ and $w = w_\nu(\rho) \in \mathsf{T}_{[\rho]}QF(S)$. If $\delta \in \left(0,\frac{\pi}{2}\right)$, then there exists $D = D(\delta,\rho_0)$ so that*

$$\mathrm{d}\ell_\gamma(w_\nu(\rho)) \leq -Ci_\delta(\gamma,\beta_+)$$

for all $\gamma \in \pi_1(S)$.

Proof. Recall that if $\mu \in \mathcal{C}(X)$, then $i_\delta(\mu,\beta_\nu) = \mu \times \beta_\nu\big(D_\delta(X)\big)$ and that $D_\delta(X)$ is compact. If $(u,v) \in D_\delta(X)$ lies in the support of $\mu \times \beta_\nu$, then u is a tangent vector to a leaf m of μ and v is a tangent vector to a leaf a of β_ν. Since Ω_ν is moderately bent, we saw in the proof of Theorem 1.3 that $\Im \cosh\sigma(a,m) < 0$. Since the intersection of the support of $\mu\times\beta_\nu$ with $D_\delta(X)$ is compact and complex length varies continuously, there exists $C > 0$ so that if $(u,v) \in D_\delta(X)$ lies in the support of $\mu\times\beta_\nu$, then $\Im\left(\cosh\sigma(a,m)\right) \leq -C$. Our formula for $d\ell_\gamma(w_\nu(\rho))$, from Theorem 3.2, immediately gives that

$$d\ell_\gamma(w_\nu(\rho)) \leq -Ci_\delta(\gamma,\beta_+)$$

for all $\gamma \in \pi_1(S)$. □

We are now ready to prove Theorem 1.4.

Theorem 1.4. *Suppose that $\rho \in \mathrm{QF}(S)$ is not fuchsian and that $w = w_+(\rho) + w_-(\rho)$. If both $\Omega_+(\rho)$ and $\Omega_-(\rho)$ are moderately bent, then there exists $K > 0$ such that*

$$\mathrm{d}\ell_\gamma(w) \le -K\ell_\gamma(\rho)$$

for all $\gamma \in \pi_1(S)$.

Proof. Recall that the bending laminations of a quasifuchsian group which is not fuchsian always bind (see, for example Bonahon-Otal [10, Proposition 4]). Fix a hyperbolic structure X on S. Lemma 5.1 implies that there exists a $\delta > 0$ and $C > 0$ so that if $\mu \in \mathcal{C}(X)$, then

$$i_\delta(\beta_+, \mu) + i_\delta(\beta_-, \mu) \ge C\ell_\mu(X).$$

Lemma 5.2 implies that there exists $D > 0$ so that

$$\mathrm{d}\ell_\gamma(w_\pm(\rho)) \le -Di_\delta(\gamma, \beta_\pm)$$

for all $\gamma \in \pi_1(S)$ and $\nu \in \{\pm\}$. Therefore,

$$\mathrm{d}\ell_\gamma(w) \le -D\Big(i_\delta(\gamma, \beta_+) + i_\delta(\gamma, \beta_-)\Big) \le -CD\ell_\gamma(X)$$

for all $\gamma \in \pi_1(S)$. Since ρ is quasifuchsian, there exists $B > 1$ so that

$$\frac{\ell_\gamma(\rho)}{B} \le \ell_\gamma(X) \le B\ell_\gamma(\rho)$$

for all $\gamma \in \pi_1(S)$. Therefore,

$$\mathrm{d}\ell_\gamma(w) \le -\frac{CD}{B}\ell_\gamma(\rho)$$

for all $\gamma \in \pi_1(S)$. □

6. Proper affine actions with quasifuchsian holonomy

In this section, we show that if both components of the domain of discontinuity of a nonfuchsian quasifuchsian representation ρ are moderately bent, then $\mathrm{Ad}\,\rho$ arises as the linear part of a proper affine action of the surface group on $\mathfrak{sl}(2,\mathbb{C})$.

In Section 6.1, we introduce the Margulis invariant of a loxodromic element in $\mathsf{PSL}(d,\mathbb{C})$ and an element of the Lie algebra $\mathfrak{sl}(d,\mathbb{C})$. We see that the Margulis invariant provides a bound on the displacement of the associated action on $\mathrm{Aff}(\mathfrak{sl}(d,\mathbb{C}))$. The reader who is only interested in the quasifuchsian case can always assume that $d = 2$ which simplifes the discussion.

In Section 6.2, we recall the theory of totally Anosov (a.k.a. Borel Anosov) representations into $\mathsf{PSL}(d,\mathbb{C})$. This is a natural setting for our work, since if a representation is totally Anosov then all infinite order elements have loxodromic image. A representation into $\mathsf{PSL}(2,\mathbb{C})$ is totally Anosov if and only if it is convex cocompact. In Section 6.3, we develop a criterion guaranteeing that a totally Anosov representation into $\mathsf{PSL}(d,\mathbb{C})$ and an associated cocycle with image in $\mathfrak{sl}(d,\mathbb{C})$ give rise to a proper affine action on $\mathrm{Aff}(\mathfrak{sl}(d,\mathbb{C}))$. In Section 6.4 we use our criterion to prove Theorem 1.2. In section 6.5 we obtain generalizations of Theorem 1.2 in other complex Lie groups.

6.1. The Margulis invariant.

The fact that we are working in $\mathsf{PSL}(d,\mathbb{C})$ means that the eigenvalues of a matrix are only well-defined up to mulitplication by a d^{th} root of unity. However, the moduli of the eigenvalues and their associated eigenspaces are still well-defined. We first develop language which makes this concrete.

If $g \in \mathsf{PSL}(d,\mathbb{C})$, we may choose a lift $\tilde{g} \in \mathsf{SL}(d,\mathbb{C})$. Let $\big(\lambda_i(\tilde{g})\big)_{i=1}^d$ denote the (generalized) eigenvalues of $\tilde{g}$ ordered so that $|\lambda_i(\tilde{g})| \ge |\lambda_{i+1}(\tilde{g})|$ for all i. Notice that the center Z_d of $\mathsf{SL}(d,\mathbb{C})$ consists of matrices of the form $r\mathrm{I}$ where r is a d^{th} root of unity. If $\tilde{g}'$ is another lift of g, then there exists $c \in Z_d$, so that

$\tilde{g} = c\tilde{g}'$. In particular, there exists a d^{th} root of unity r, so that $\lambda_i(\tilde{g}) = r\lambda_i(\tilde{g}')$ for all i. So we may define the *Jordan projection* $\mu : \mathsf{PSL}(d,\mathbb{C}) \to \mathfrak{a}^+$ where

$$\mathfrak{a}^+ = \left\{ \operatorname{diag}(a_1,\ldots,a_d) : \ a_1,\ldots,a_d \in \mathbb{R}, \ a_i \geq a_{i+1} \forall \ i \text{ and } \sum_{i=1}^d a_i = 0 \right\} \text{ and } \mu(g) = (\log|\lambda_1(\tilde{g})|,\ldots,\log|\lambda_d(\tilde{g})|).$$

We say that g is loxodromic if any (hence every) lift $\tilde{g}$ is loxodromic, i.e. is diagonalizable over $\mathbb{C}$ and its eigenvalues have distinct moduli. If g is loxodromic, then there exists $\tilde{\psi} \in \mathsf{PSL}(d,\mathbb{C})$ that conjugates $\tilde{g}$ to a diagonal matrix $\operatorname{diag}(b_1,\ldots,b_d)$ with decreasing in modulus entries, i.e. $|b_i| > |b_{i+1}|$ for all i. Then $\tilde{\psi}$ projects to $\psi \in \mathsf{PSL}(d,\mathbb{C})$ and we will say that ψ conjugates g into standard form.

If g is loxodromic, let $L_i(g)$ denote the one-dimensional eigenspace of $\tilde{g}$ with eigenvalue $\lambda_i(\tilde{g})$. The eigenspace $L_i(g)$ is well-defined since any two lifts differ by an element of Z_d and Z_d acts trivially on $\mathbb{P}(\mathbb{C}^d)$. Let $E_i(g)$ be the i-dimensional space spanned by the first i eigenspaces, i.e. $E_i(g) = \langle L_1(g),\ldots,L_i(g)\rangle$. The flag $E(g) = \big(E_i(g)\big)_{i=1}^{d-1}$ is the attracting flag of g. Notice that if ψ is an element conjugating g into standard form and

$$E^0 = \{\langle e_1\rangle, \langle e_1, e_2,\rangle, \ldots, \langle e_1,\ldots,e_{d-1}\rangle\}$$

then $E(g) = \psi(E^0)$. The repelling flag of g is given by $F(g) = \big(F_i(g)\big)_{i=1}^{d-1}$ where $F_i(g) = \langle L_{d-i+1},\ldots,L_d(g)\rangle$. Notice that $E(g) = F(g^{-1})$.

The Lie algebra $\mathfrak{sl}(d,\mathbb{C})$ of $\mathsf{PSL}(d,\mathbb{C})$ contains a subalgebra $\mathfrak{h}$ consisting of traceless diagonal matrices (over $\mathbb{C}$). More precisely,

$$\mathfrak{h} = \left\{\operatorname{diag}(z_1,\ldots,z_d) : \sum z_i = 0\right\}.$$

The adjoint action $\operatorname{Ad}(g) : \mathfrak{sl}(d,\mathbb{C}) \to \mathfrak{sl}(d,\mathbb{C})$ is given $\operatorname{Ad}(g)(v) = \tilde{g}v\tilde{g}^{-1}$. (Notice that again the choice of lift is irrelevant since any two lifts differ by an element of Z_d and $\operatorname{Ad}(c)$ is the identity map for any $c \in Z_d$.) If g is loxodromic, $\operatorname{Ad}(g)$ is also diagonalizable (although it may have eigenvalue coincidences). If ψ conjugates g into standard form, then its 1-eigenspace is given by

$$V^0(g) := \operatorname{Fix}(\operatorname{Ad}(g)) = \operatorname{Ad}(\psi^{-1})\mathfrak{h}.$$

We can consider the projection

$$\pi_g^0 : \mathfrak{sl}(d,\mathbb{C}) \to V^0(g)$$

whose kernel $W(g)$ is $\operatorname{Ad}(g)$-invariant. More explicitly $W(g)$ is the sum of the eigenspaces associated to all the eigenvalues but 1.

Margulis [51] originally defined the Margulis invariant for pairs in $\mathrm{SO}(1,2) \ltimes \mathbb{R}^3$. The following generalization is due to Smilga [61].

Definition 6.1. If $g \in \mathsf{PSL}(d,\mathbb{C})$ and $x \in \mathfrak{sl}(d,\mathbb{C})$ the *Margulis invariant* of (g,x) is

$$\mathsf{m}(g,x) := \operatorname{Ad}(\psi)\big(\pi_g^0(x)\big) \in \mathfrak{h},$$

i.e. we project x onto the fixed set $V^0(g)$ of $\operatorname{Ad}(g)$ and move it into $\mathfrak{h}$ via $\operatorname{Ad}(\psi)$.

Since any two elements ψ and ψ' which conjugate g into standard form satisfy $\psi'\psi^{-1} \in \exp\mathfrak{h}$ and $\operatorname{Ad}(\exp\mathfrak{h})$ acts trivially on $\mathfrak{h}$, the Margulis invariant $\mathsf{m}(g,x)$ is independent of the choice of ψ. Moreover,

- $\mathsf{m}(g,x)$ is invariant under conjugation by an element of $\mathsf{PSL}(d,\mathbb{C}) \ltimes \mathfrak{sl}(d,\mathbb{C})$, and
- $\mathsf{m}\big((g,x)^{-1}\big) = \mathrm{i}\big(\mathsf{m}(g,x)\big)$, where i is the opposition involution of $\mathfrak{h}$.

In our proof it will be useful to choose ψ efficiently.

Lemma 6.2. (Bochi-Potrie-Sambarino [8, Proposition 7.2]) *Given $d \in \mathbb{N}$, there exists $K > 1$ so that if $g \in \mathsf{PSL}(d,\mathbb{C})$ is loxodromic, one may choose an element ψ which conjugates g into standard form so that*

$$\|\psi\| \min_{i \in [\![1,d-1]\!]} \angle(E_i(g), F_{d-i}(g)) \in (1/K, K)$$

where $\|\psi\|$ is the standard operator norm of ψ.

Remark 6.3 (When $d=2$). If $g \in \mathsf{PSL}(2,\mathbb{C})$ is loxodromic, then there exists $\psi \in \mathsf{PSL}(2,\mathbb{C})$ so that $\psi g \psi^{-1} = \pm \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ where $|\lambda| > 1$. So,

$$V^0(g) := \mathrm{Fix}\,(\mathrm{Ad}(g)) = \mathrm{Ad}(\psi^{-1})\mathfrak{h} = \left\langle \psi^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \psi \right\rangle.$$

The other two eigenspaces of $\mathrm{Ad}(g)$ are

$$\left\langle \psi^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \psi \right\rangle \quad \text{and} \quad \left\langle \psi^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \psi \right\rangle$$

with eigenvalues λ^2 and λ^{-2} respectively. Moreover, $E(g)$ is the attracting eigenline of g and $F(g)$ is the repelling eigenline. Notice that the opposition involution is the identity map when $d=2$, so $\mathsf{m}\big((g,x)^{-1}\big) = \mathsf{m}(g,x)$.

An element $(g,x) \in \mathsf{PSL}(d,\mathbb{C}) \ltimes \mathfrak{sl}(d,\mathbb{C})$ induces an affine transformation

$$\mathsf{F}_{(g,x)} : \mathfrak{sl}(d,\mathbb{C}) \to \mathfrak{sl}(d,\mathbb{C}) \quad \text{by} \quad \mathsf{F}_{(g,x)}v = \mathrm{Ad}(g)v + x.$$

When the angles between the attracting/repelling flags of g are uniformly bounded below, then the displacement of $\mathsf{F}_{(g,x)}$ is controlled by the Margulis invariant:

Lemma 6.4 (Margulis invariant as least displacement)**.** *Given $C>0$ and $d \in \mathbb{N}$, there exists $c>0$ such that if $g \in \mathsf{PSL}(d,\mathbb{C})$ is loxodromic and*

$$\min_{i \in [\![1,d-1]\!]} \angle(E_i(g), F_{d-i}(g)) \geq C,$$

then

$$\big\|\mathsf{F}_{(g,x)}v - v\big\| \geq c\|\mathsf{m}(g,x)\|.$$

for any $x, v \in \mathfrak{sl}(d,\mathbb{C})$.

Proof. Since the angles of the attracting/repelling flags of g are bounded below by C, the angles between the spaces in the decomposition

$$\mathfrak{sl}(d,\mathbb{C}) = V^0(g) \oplus W(g)$$

are uniformly bounded below. Thus there exists $k>0$ (depending only on C and d) such that $\|u\| \geq \|\pi^0_g(u)\|$ for every $u \in \mathfrak{sl}(d,\mathbb{C})$. In particular, if $u = \mathsf{F}_{(g,x)}v - v$ we obtain

$$\|\mathsf{F}_{(g,x)}v - v\| \geq k\big\|\pi^0_g(\mathsf{F}_{(g,x)}v - v)\big\|. \tag{2}$$

Now observe that, since $\mathrm{Ad}(g)|_{V^0(g)} = \mathrm{id}$, we have

$$\pi^0_g\big(\mathsf{F}_{(g,x)}v - v\big) = \pi^0_g\big(\mathrm{Ad}(g)v + x - v\big) = \pi^0_g(x),$$

It follows from the definition of m that

$$\mathrm{Ad}(\psi)\big(\pi^0_g\big(\mathsf{F}_{(g,x)}v - v\big)\big) = \mathsf{m}(g,x).$$

for all $v \in \mathfrak{sl}(d,\mathbb{C})$, where ψ is an element conjugating g into standard form. By Lemma 6.2 we can chose ψ so that $\|\psi\| < K$, where K only depends on C and d. Therefore

$$\|\pi^0_g(\mathsf{F}_{(g,x)}v - v)\| \geq K^{-1}\big\|\mathrm{Ad}(\psi)(\pi^0_g(\mathsf{F}_{(g,x)}v - v)\big\| = K^{-1}\|\mathsf{m}(g,x)\|,$$

which combining with (2) finishes the proof. □

Let

$$\sigma_1(g) \geq \cdots \geq \sigma_d(g)$$

be the singular values of g (with respect to the standard Hermitian metric on $\mathbb{C}^d$). The *Cartan projection*

$$\kappa : \mathsf{PSL}(d,\mathbb{C}) \to \mathfrak{a}^+ \quad \text{is given by} \quad \kappa(g) = \mathrm{diag}(\log \sigma_i(g)).$$

We observe that when the angles between the attracting/repelling flags of g are uniformly bounded below then one can bound the ratio of the top eigenvalue and the top singular value.

Lemma 6.5. *Given $C > 0$ and $d \in \mathbb{N}$, there exists $b > 0$ such that if $g \in \mathsf{PSL}(d, \mathbb{C})$ is loxodromic and*

$$\min_{i \in [\![1,d-1]\!]} \angle(E_i(g), F_{d-i}(g)) \geq C,$$

then

$$\left|\omega_1(\mu(g)) - \log \sigma_1(g)\right| < b$$

where $\omega_1(\mu(g)) = \log |\lambda_1(\tilde{g})|$.

Proof. Lemma 6.2 implies that there exists $K > 1$, which depends only on C and d, so that $\psi g \psi^{-1}$ is in standard form and $||\psi|| < K$. Now notice that $|\omega_1(\mu(g))| = \|\psi g \psi^{-1}\|$ and

$$\sigma_1(g) = ||g|| \leq \|\psi^{-1}\| \cdot \|\psi g \psi^{-1}\| \cdot \|\psi\| \leq \|\psi\|^d \cdot \omega_1(\mu(g)) \leq K^d |\mu_1(g)|$$

since $\|\psi^{-1}\| \leq \|\psi\|^{d-1}$. So, our result hold with $b = d \log K$. □

6.2. **Totally Anosov representations in $\mathsf{PSL}(d, \mathbb{C})$.** Anosov representations were introduced by Labourie [48] and have been a very active topic of research ever since. The surveys by Kassel [41] and Wienhard [65] give an overview of the topic so we will focus on the facts that we need.

Let Γ be a finitely generated group and let $|\,|$ denote the word-length on Γ associated to some fixed finite generating set. A representation $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{C})$ is *totally Anosov* if there exist positive C, c such that for every $k \in [\![1, d-1]\!]$ and $\gamma \in \Gamma$ one has

$$\frac{\sigma_{k+1}(g)}{\sigma_k(g)} \leq C e^{-c|\gamma|}.$$

This characterization of totally Anosov representations was established by Kapovich-Leeb-Porti [40] and Bochi-Potrie-Sambarino [8]. Totally Anosov representations into $\mathsf{PSL}(d, \mathbb{C})$ are often called Borel Anosov, since they are Anosov with respect to the Borel subgroup of $\mathsf{PSL}(d, \mathbb{C})$

The space of totally Anosov representations will be denoted by

$$\mathfrak{A}_{\Delta}(\Gamma, \mathsf{PSL}(d, \mathbb{C})).$$

It is an open subset of the character variety (see [48] and [31]). Moreover, if Γ admits a totally Anosov representation, then Γ is necessarily word-hyperbolic (see [40]) and there exists a continuous equivariant map $\xi : \partial\Gamma \to \mathcal{F}(\mathbb{C}^d)$ such that if $x, y \in \partial\Gamma$ are distinct, then $\xi(x)$ and $\xi(y)$ are transverse flags. Moreover, for every infinite order element $\gamma \in \Gamma$, its image $\rho(\gamma)$ is loxodromic and its attracting/repelling flags are, respectively, $\xi(\gamma_+)$ and $\xi(\gamma_-)$ (see [48] and [31]). In particular, totally Anosov representations into $\mathsf{PSL}(d, \mathbb{C})$ are natural generalizations of convex cocompact representations into $\mathsf{PSL}(d, \mathbb{C})$.

6.3. **A criterion for proper affine actions.** If $\rho : \Gamma \to \mathsf{PSL}(d, \mathbb{C})$ is totally Anosov, a *cocycle* for $\mathrm{Ad}\,\rho$ is a map $\mathsf{u} : \Gamma \to \mathfrak{sl}(d, \mathbb{C})$ such that for every $\gamma, \eta \in \Gamma$ one has

$$\mathsf{u}(\gamma\eta) = \mathsf{u}(\gamma) + \mathrm{Ad}\,\rho(\gamma)\mathsf{u}(\eta).$$

This equation implies that the map

$$(\rho, \mathsf{u}) : \Gamma \to \mathsf{PSL}(d, \mathbb{C}) \ltimes \mathfrak{sl}(d, \mathbb{C}) = \mathrm{Aff}(\mathfrak{sl}(d, \mathbb{C}))$$

is a representation and we consider the associated group of affine transformations $\mathsf{F}_{(\rho,\mathsf{u})}(\Gamma)$ given by

$$\mathsf{F}_{(\rho,\mathsf{u})}(\gamma) = \mathsf{F}_{(\rho(\gamma),\mathsf{u}(\gamma))}.$$

The most common way to construct a cocycle is to consider a smooth family $\{\rho_t : \Gamma \to \mathsf{PSL}(d, \mathbb{C})\}$ and let

$$\mathsf{u}(\gamma) = \frac{d}{dt}\Big|_{t=0} \rho_t(\gamma)\rho_0(\gamma)^{-1} \in \mathfrak{sl}(d, \mathbb{C}).$$

It is easy to check that u is a cocycle for $\mathrm{Ad}\,\rho$.

The *normalized Margulis spectra* is the set

$$\mathrm{MS}(\rho, \mathsf{u}) = \overline{\left\{ \frac{\mathsf{m}(\rho(\gamma), \mathsf{u}(\gamma))}{\omega_1(\mu(\rho(g)))} : \gamma \in \Gamma \text{ has infinite order} \right\}}$$

where $\omega_1(\mu(\rho(g))) = \log |\lambda_1(\widetilde{\rho(g)})|$. It is always a compact convex subset of $\mathfrak{h}$.

The following is a particular case of a result of Kassel-Smilga [42], see also Ghosh [32]. We include a proof for completeness.

Proposition 6.6 (Properness Criteria)**.** *Suppose that* $\rho : \Gamma \to \mathsf{PSL}(d,\mathbb{C})$ *is totally Anosov and* $\mathsf{u} : \Gamma \to \mathfrak{sl}(d,\mathbb{C})$ *is a cocycle for* ρ*. If* $0 \notin \mathrm{MS}(\rho,\mathsf{u})$*, then* $\mathsf{F}_{(\rho,\mathsf{u})}(\Gamma)$ *acts properly discontinuously on* $\mathfrak{sl}(d,\mathbb{C})$*.*

Proof. We follow the outline of Smilga [61]. We simplify notation by letting $\mathsf{F} := \mathsf{F}_{(\rho,\mathsf{u})}$ throughout the proof. Since $0 \notin \mathrm{MS}(\rho,\mathsf{u})$, there exists $a > 0$ so that

$$\|m(\rho(\gamma),\mathsf{u}(\gamma))\| \geq a\omega_1(\mu(\rho(\gamma)))$$

for all $\gamma \in \Gamma$.

Guichard and Wienhard [31, Thm. 5.10] used work of Abels, Margulis and Soifer [1] to show that if ρ is totally Anosov, then there exists a finite subset $A \subset \Gamma$ and $C > 0$ so that if $\gamma \in \Gamma$, then there exists $\alpha \in A$ so that $\rho(\alpha\gamma)$ is loxodromic and

$$\min_{i \in [\![1,d-1]\!]} \angle(E_i(\alpha\gamma), F_{d-i}(\alpha\gamma)) \geq C. \tag{3}$$

Lemmas 6.4 and 6.5 then imply that there exists $b, c > 0$ so that for all $\gamma \in \Gamma$ there exists $\alpha \in A$ so that

$$\|\mathsf{F}(\alpha\gamma)v - v\| \geq c\big\|\mathsf{m}\big(\rho(\alpha\gamma),\mathsf{u}(\alpha\gamma)\big)\big\| \text{ for all } v \in \mathfrak{sl}(d,\mathbb{C}) \quad \text{and} \quad |\omega_1(\mu(\rho(\alpha\gamma)) - \log\sigma_1(\rho(\alpha\gamma)| < b. \tag{4}$$

Let $K \subset \mathfrak{sl}(d,\mathbb{C})$ be a compact set and $S = \{\gamma \mid \mathsf{F}(\gamma)K \cap K \neq \emptyset\}$. We consider the compact set

$$K' = \bigcup_{\alpha \in A} \mathsf{F}(\alpha)(K).$$

If $\gamma \in S$, then we can choose $\alpha \in A$ so that $\rho(\alpha\gamma)$ is loxodromic and satisfies Equation (4). Since $\gamma \in S$, we see that $\mathsf{F}(\alpha\gamma)(K) \cap \mathsf{F}(\alpha)(K) \neq \emptyset$, so $\mathsf{F}(\alpha\gamma)K \cap K' \neq \emptyset$. Therefore, combining Equations (3) and (4), we see that

$$\omega_1(\mu(\rho(\alpha\gamma))) \leq \frac{1}{ac} \inf_{v \in \mathfrak{sl}(d,\mathbb{C})} \|\mathsf{F}(\alpha\gamma)v - v\| \leq \frac{1}{ac}\mathrm{diam}(K \cup K') = C_1,$$

so, again by Equation (4),

$$\log\sigma_1(\rho(\alpha\gamma)) < b + C_1.$$

If we set $d = \max_{\alpha \in A} \log\sigma_1(\alpha)$, then we conclude that

$$\log\sigma_1(\rho(\gamma)) < b + C_1 + d.$$

Since there are only finitely many elements of Γ such that $\log\sigma_1(\rho(\gamma)) < b + C_1 + d$, we conclude that S is finite and so F is proper.

□

Remark: If $d = 2$ and ρ is quasifuchsian it is easy to use a ping-pong argument to produce two elements α_1 and α_2 so that if $\gamma \in \rho(\pi_1(S))$, then the attracting and repelling fixed points of $\alpha_i\gamma$ are uniformly separated for either $i = 1$ or $i = 2$.

6.4. **Proper affine actions on $\mathfrak{sl}(2,\mathbb{C})$.** In order to apply our Properness criterion and prove Theorem 1.2, we must be able to compute the Margulis invariant. We first introduce some notation.

If $\tilde{g} \in \mathsf{SL}(d,\mathbb{C})$ is loxodromic, we define

$$\lambda : \mathsf{SL}(d,\mathbb{C}) \to \mathsf{SL}(d,\mathbb{C}) \quad \text{by} \quad \lambda(\tilde{g}) = \mathrm{diag}\big(\lambda_1(\tilde{g}),\ldots,\lambda_d(\tilde{g})\big).$$

Recall that λ is an analytic function on the open set of loxodromic elements in $\mathsf{SL}(d,\mathbb{C})$. If $g \in \mathsf{PSL}(d,\mathbb{C})$ is loxodromic and $\dot{g} \in \mathsf{T}_g\mathsf{PSL}(d,\mathbb{C})$, let $\{g_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth family of loxodromic elements in $\mathsf{PSL}(d,\mathbb{C})$ so that $g_0 = g$ and $\frac{d}{dt}\big|_{t=0} g_t = \dot{g}$. One can lift $\{g_t\}_{t\in(-\epsilon,\epsilon)}$ to a smooth family $\{\tilde{g}_t\}_{t\in(-\epsilon,\epsilon)}$ of loxodromic elements of $\mathsf{SL}(d,\mathbb{C})$. We abuse notation by defining

$$\big(\mathrm{d}\lambda(\dot{g})\big)\lambda(g)^{-1} = \left(\mathrm{d}\lambda\left(\frac{d}{dt}\Big|_{t=0}\tilde{g}_t\right)\right)\lambda\big(\tilde{g}_0^{-1}\big) \in \mathfrak{h}.$$

Since any two lifts differ by multiplication by an element of Z_d, this is well-defined. We also let

$$x_g := \dot{g}g^{-1} \in \mathfrak{sl}(d,\mathbb{C}).$$

The following computation was first performed by Goldman and Margulis [39] when $\mathsf{G} = \mathsf{SO}(2,1)$. We give a version which appears in Sambarino [59, Cor. 8.3]. Similar computations can also be found in Ghosh [32, 33], Kassel-Smilga [42] and Danciger-Zhang [23]. We include a proof for completeness.

Proposition 6.7. *If $g \in \mathsf{PSL}(d,\mathbb{C})$ is loxodromic and $\dot g \in \mathsf{T}_g\mathsf{PSL}(d,\mathbb{C})$, then*
$$\mathsf{m}(g,x_g) = \big(\mathrm{d}\lambda(\dot g)\big)\lambda(g)^{-1}$$
In particular,
$$\Re\Big(\mathsf{m}(g,x_g)\Big) = \mathrm{d}\mu(\dot g).$$

Proof. Let $\{g_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth family of loxodromic elements in $\mathsf{PSL}(d,\mathbb{C})$ so that $g_0 = g$ and $\frac{d}{dt}\big|_{t=0} g_t = \dot g$. Lift $\{g_t\}_{t\in(-\epsilon,\epsilon)}$ to a smooth family of loxodromic elements of $\mathsf{SL}(d,\mathbb{C})$ which we will further abuse notation by continuing to call $\{g_t\}_{t\in(-\epsilon,\epsilon)}$. We may choose a smooth family $\{\psi_t\}_{t\in(-\epsilon,\epsilon)}$ in $\mathsf{SL}(d,\mathbb{C})$ so that $\mu(\gamma_t) = \psi_t g_t \psi_t^{-1}$ for all t. Let $\psi = \psi_0$ and $\psi_t = \psi\phi_t$, so $\phi_0 = \mathrm{Id}$ and $\frac{d}{dt}\big|\phi_t = \dot\phi_0 \in \mathfrak{sl}(d,\mathbb{C})$. Differentiating the expression $\lambda(g_t) = \psi(\phi_t g_t \phi_t^{-1})\psi^{-1}$ gives
$$\mathrm{d}\lambda(\dot g) = \psi(\phi_0 \dot g \phi_0^{-1} + \dot\phi_0 g_0 - g_0\dot\phi_0)\psi^{-1} = \psi(\dot g + \dot\phi_0 g_0 - g_0\dot\phi_0)\psi^{-1},$$
so
$$\mathrm{d}\lambda(\dot g)\lambda(g)^{-1} = \psi(\dot g + \dot\phi_0 g_0 - g_0\dot\phi_0)g_0^{-1}\psi^{-1} = \psi(\dot g g_0^{-1} + \dot\phi_0 - g_0\dot\phi_0 g_0^{-1})\psi^{-1}.$$
Therefore
$$\mathrm{d}\lambda(\dot g)\lambda(g)^{-1} = \mathrm{Ad}(\psi)(x_g + \dot\phi_0 - \mathrm{Ad}(g)(\dot\phi_0)).$$

Recall that the map π^0_g is projection onto the 1-eigenspace $V^0(g)$ of $\mathrm{Ad}(g)$ parallel to the sum $W(g)$ of the other eigenspaces. Thus
$$\pi^0_{\psi g\psi^{-1}} = \mathrm{Ad}(\psi)\circ\pi^0_g\circ\mathrm{Ad}(\psi^{-1}).$$
Since $\psi g\psi^{-1}$ is diagonal and loxodromic, $\pi^0_{\psi g\psi^{-1}} = \pi^0$ where $\pi^0 : \mathfrak{sl}(d,\mathbb{C}) \to \mathfrak{h}$ is the projection to the diagonal obtained by changing all the non-diagonal entries to 0. Therefore
$$\pi^0\circ\mathrm{Ad}(\psi) = \mathrm{Ad}(\psi)\circ\pi^0_g.$$
Applying π^0 to both sides gives
$$\mathrm{d}\lambda(\dot g)\lambda(g)^{-1} = \pi^0(d\lambda(\dot g)\lambda(g)^{-1}) = \mathrm{Ad}(\psi)\Big(\pi^0_g\big(x_g + \dot\phi_0 - \mathrm{Ad}(g)(\dot\phi_0)\big)\Big).$$
Since $\pi^0_g\circ\mathrm{Ad}(g) = \pi^0_g$, we conclude that
$$\mathrm{d}\lambda(\dot g)\lambda(g)^{-1} = \mathrm{Ad}(\psi)(\pi^0_g(x_g)) = \mathsf{m}(g,x_g).$$

Our formula for $\Re\Big(\mathsf{m}(g,x_g)\Big)$ follows from the elementary fact that if $(z_t)_{t\in(-\varepsilon,\varepsilon)}$ is a smooth curve in $\mathbb{C}^*$, then $\Re\big(\dot z/z\big) = \frac{\partial}{\partial t}\big|_{t=0}\log|z_t|$. ☐

With this computation in hand, we can combine Theorem 1.4 and Theorem 6.8 to establish Theorem 1.2.

Theorem 1.2. *If $[\rho] \in \mathrm{QF}(S)$ is not fuchsian and both $\Omega_+(\rho)$ and $\Omega_-(\rho)$ are moderately bent, then there exists a representation $\sigma : \pi_1(S) \to \mathrm{Aff}(\mathfrak{sl}(2,\mathbb{C}))$ whose linear part is $\mathrm{Ad}(\rho)$ so that $\sigma(\pi_1(S))$ acts properly discontinuously on $\mathfrak{sl}(2,\mathbb{C})$.*

Proof. Let $w = w(\rho) = w_+(\rho) + w_-(\rho)$. Let $\{\rho_t\}_{t\in(-\epsilon,\epsilon)}$ be a smooth family in $\mathrm{Hom}\big(\pi_1(S),\mathsf{PSL})2,\mathbb{C})\big)$ such that $\rho_0 = \rho$ and
$$\frac{d}{dt}\Big|_{t=0}\rho_t = w$$
Let u be the cocycle for $\mathrm{Ad}\,\rho$ given by $\mathsf{u}(g) = \frac{d}{dt}\Big|_{t=0}\rho_t(g)\rho_0(g)^{-1}$. Theorem 1.4 implies that there exists a $K > 0$ so that
$$\mathrm{d}\ell_\gamma(w) \le -K\ell_\gamma(\rho)$$

for all $\gamma \in \pi_1(S)$. Since $\ell_\rho(\gamma) = 2\log(\mu_1(\gamma))$, Proposition 6.7 implies that

$$\Re\Big(\mathsf{m}(\rho(g), \mathsf{u}(g))\Big) = \Big(\mathrm{d}\ell_\gamma(\mathsf{u}), -\mathrm{d}\ell_\gamma(\mathsf{u})\Big)$$

and so the first coordinate of $\frac{\Re\left(\mathsf{m}(\rho(g)\mathsf{u}(g))\right)}{\omega_1(\rho(g))}$ is always less than $-K$. Therefore, $0 \notin \mathrm{MS}(\rho, \mathsf{u})$, so our Properness Criterion, Proposition 6.6, implies that the action $F_{\rho,\mathsf{u}}$ is proper. □

6.5. **Proper affine actions on other complex Lie groups.** We first observe that one may generalize the proof of Theorem 1.2 into the setting of totally Anosov representations into $\mathsf{PSL}(d, \mathbb{C})$. Let $\mathfrak{a}$ denote the real part of $\mathfrak{h}$ (i.e. the matrices in $\mathfrak{h}$ with real entries) and let $\mathfrak{a}^*$ be its dual space. Suppose that Γ is a finitely generated group. If $\varphi \in \mathfrak{a}^*$ and $\gamma \in \Gamma$, consider the function

$$\varphi^\gamma : \mathrm{Hom}\big(\Gamma, \mathsf{PSL}(d, \mathbb{C})\big) \to \mathfrak{a} \quad \text{given by} \quad \varphi^\gamma : \rho \mapsto \varphi\big(\mu\big(\rho(\gamma)\big)\big).$$

If $(\rho_t)_{t\in(-\varepsilon,\varepsilon)} : \Gamma \to \mathsf{PSL}(d, \mathbb{C})$ is a smooth curve in $\mathfrak{X}\big(\Gamma, \mathsf{PSL}(d, \mathbb{C})\big)$ so that ρ_0 is totally Anosov, then φ^γ is differentiable at 0, whenever $\gamma \in \Gamma$ has infinite order.

Corollary 6.8. *Let $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ be a smooth curve in $\mathrm{Hom}\big(\Gamma, \mathsf{PSL}(d, \mathbb{C})\big)$ such that $\rho = \rho_0$ is totally Anosov and let $\mathsf{u} : \Gamma \to \mathfrak{g}$ be the cocycle given by $\mathsf{u}(g) = \frac{d}{dt}\Big|_{t=0} \rho_t(g)\rho_0(g)^{-1}$. If there exists $\varphi \in \mathfrak{a}^*$ such that*

$$\mathrm{d}\varphi^\gamma(\mathsf{u}) \geq \omega_1(\mu(\rho(\gamma)))$$

for all $\gamma \in \Gamma$, then $\mathsf{F}_{(\rho,\mathsf{u})}(\Gamma)$ acts properly discontinuously on $\mathfrak{sl}(d, \mathbb{C})$ is proper.

Proof. By Proposition 6.6 it suffices to show that $0 \notin \mathrm{MS}(\rho, \mathsf{u})$. For $\gamma \in \Gamma$ we have, by Proposition 6.7 and our assumption imply that

$$\varphi\big(\Re\big(\mathsf{m}(\rho(\gamma), \mathsf{u}(\gamma))\big)\big) = \mathrm{d}\varphi^\gamma(\mathsf{u}) \geq \omega_1(\mu(\rho(\gamma))).$$

Thus $\varphi\Big(\Re\big(\mathrm{MS}(\rho, \mathsf{u})\big)\Big) \subset [1, \infty)$, which implies that $0 \notin \mathrm{MS}(\rho, \mathsf{u})$. □

We now consider other complex simple Lie groups. If G is a complex simple Lie group with Lie algebra $\mathfrak{g}$, the concepts from the previous sections, such as the Cartan algebra $\mathfrak{h}$, the Jordan projection $\mu : \mathsf{G} \to \mathfrak{a} = \Re(\mathfrak{h})$, and the Cartan projection $\kappa : \mathsf{G} \to \mathfrak{a} = \Re(\mathfrak{h})$ extend to this setting. Finally the Margulis invariant $\mathsf{m}(g, x) \in \mathfrak{h}$ of $(g, x) \in \mathsf{G} \ltimes \mathfrak{g}$, can be extended to this more general setting. It will suffice for our purposes to recall that (after identifying $\mathsf{G} \ltimes \mathfrak{g}$ with TG) for $(g, x) = \dot{g} \in \mathsf{T}_g\mathsf{G}$,

$$\Re\big(\mathsf{m}(g, x)\big) = \mathrm{d}\mu(\dot{g})$$

(see [59, Corollary 8.3]). We refer the reader to the book of Benoist and Quint [5] for the standard Lie algebra and Lie group concepts and to Smilga [61] for a discussion of the Margulis invariant in this more general context.

In this setting it is convenient to consider the *first fundamental restricted weight* $\varpi_1 \in \mathfrak{a}^*$ of G. The number $\varpi_1(\mu(g))$ replaces $\omega_1(\mu(g))$ (and coincides with it when $\mathsf{G} = \mathsf{PSL}(d, \mathbb{C})$). We say that a representation $\rho : \Gamma \to \mathsf{G}$ is totally Anosov if it is Anosov with respect to a minimal parabolic subgroup of G (see [31]). Let $\mathfrak{A}_\Delta(\Gamma, \mathsf{G})$ be the subspace of the character variety $\mathfrak{X}(\Gamma, \mathsf{G})$ consisting of totally Anosov representations.

For a homomorphism $(\rho, u) : \Gamma \to \mathsf{G} \ltimes \mathfrak{g}$ the normalized Margulis spectrum is the subset of $\mathfrak{h}$,

$$\mathrm{MS}(\rho, \mathsf{u}) = \overline{\left\{ \frac{\mathsf{m}(\rho(\gamma), \mathsf{u}(\gamma))}{\varpi_1\big(\mu(\rho(\gamma))\big)} : \gamma \in \Gamma \text{ has infinite order } \right\}}.$$

We recall that the Margulis spectrum varies continuously.

Proposition 6.9 (Sambarino [59, Lemma 2.34])**.** *The map $(\rho, \mathsf{u}) \mapsto \mathrm{MS}(\rho, \mathsf{u})$ is continuous on the open subset of $\mathcal{C}(\Gamma, \mathsf{G})$ where $\rho \in \mathfrak{A}_\Delta(\Gamma, \mathsf{G})$ (where the image lies in the space of compact subsets of $\mathfrak{h}$ with the Hausdorff topology).*

We are now ready to prove part (1) of Corollary 1.7. Recall that $\mathcal{C}(S, \mathsf{G})$ is the space of pairs (ρ, u) where $\rho : \pi_1(S) \to \mathsf{G}$ is a representation and u is a cocycle for $\mathrm{Ad}\,\rho$.

Theorem 6.10. *Let* G *be a complex simple Lie group with Lie algebra* $\mathfrak{g}$*, then the subset of* $\mathcal{C}(S,\mathsf{G})$ *consisting of pairs* (ρ,u) *so that* $F_{\rho,\mathsf{u}}(\pi_1(S))$ *act properly on* $\mathfrak{g}$ *has non-empty interior.*

Proof. We first give the proof in the case that $\mathsf{G}=\mathrm{Inn}(\mathfrak{g})$ is the group of inner autmorphisms of $\mathfrak{g}$. Theorem 1.2 implies there exists $\rho\in\mathrm{QF}(S)$ and a cocycle u for $\mathrm{Ad}\,\rho$ such that $0\notin\Re(\mathrm{MS}(\rho,\mathsf{u}))$. Kostant defined an embedding $\tau:\mathsf{PSL}(2,\mathbb{C})\to\mathrm{Inn}(\mathfrak{g})$, called the principal embedding, which is unique up to conjugation. If $\mathsf{G}=\mathsf{PSL}(d,\mathbb{C})$, the principle embedding is the irreducible embedding (see Kostant [45]). The principal embedding $\tau:\mathsf{PSL}(2,\mathbb{C})\to\mathrm{Inn}(\mathfrak{g})$ has the property that

$$\varpi_1(\mu(\tau(g)))=c_{\mathfrak{g}}\omega_1(\mu(g))$$

for all $g\in\mathsf{PSL}(2,\mathbb{C})$, where $c_{\mathfrak{g}}>0$ is an explicit constant ($c_{\mathfrak{g}}=d-1$ when $\mathfrak{g}=\mathfrak{sl}(d,\mathbb{C})$). The principal embedding τ induces an embedding

$$\mathrm{d}\tau:\mathfrak{sl}(2,\mathbb{C})\to\mathfrak{g}\quad\text{so that}\quad\varpi_1\big(\Re(\mathrm{MS}(\tau(\rho),\mathrm{d}\tau\circ\mathsf{u})\big)=\omega_1(\Re(\mathrm{MS}(\rho,\mathsf{u}))),$$

so $0\notin\Re(\mathrm{MS}(\tau(\rho),\mathrm{d}\tau(\mathsf{u})))$.

Since the map $(\eta,\vec{v})\mapsto\mathrm{MS}(\eta,\vec{v})$ is continuous when η is totally Anosov and $0\notin\varpi_1(\Re(\mathrm{MS}(\tau(\rho),\mathrm{d}\tau(\mathsf{u}))))$, we conclude that the same holds for nearby $(\eta,\vec{v})$. In particular, $0\notin\mathrm{MS}(\eta,\vec{v})$ and thus the analogue of Proposition 6.8 (see [42] or [59, Prop. 7.2]) shows that $\mathsf{F}_{(\eta,\vec{v})}(\pi_1(S))$ acts properly discontinuously on $\mathfrak{g}$.

If G is not $\mathrm{Inn}(\mathfrak{g})$, then there is a covering map $p:\mathsf{G}\to\mathrm{Inn}(\mathfrak{g})$ and τ lifts to a homomorphism $\tilde{\tau}:\mathsf{SL}(2,\mathbb{C})\to\mathsf{G}$ such that $\varpi_1(\mu(\tilde{\tau}(g)))=c_{\mathfrak{g}}\omega_1(\mu(g))$. Recall that every quasifuchsian representation $\rho:\pi_1(S)\to\mathsf{PSL}(2,\mathbb{C})$ lifts to a representation $\tilde{\rho}:\pi_1(S)\to\mathsf{SL}(2,\mathbb{C})$ (see Culler [19]). Since $\mathrm{Ad}\,\tilde{\rho}=\mathrm{Ad}\,\rho$ and $\mathrm{d}\tilde{\tau}\circ u=\mathrm{d}\tau\circ u$, we may complete the proof in this case using the same argument. □

7. Proper actions on the group manifold

In this section, we study actions on the group manifold of a Lie group via left and right multiplication. The final goal of the section is to establish part (2) of Corollary 1.7.

The action of $\mathsf{G}\times\mathsf{G}$ on G via $(g,h)x=gxh^{-1}$ is transitive and the stabilizer of $e\in\mathsf{G}$ is the diagonal embedding

$$\mathrm{Diag}(\mathsf{G})=\{(g,g):g\in\mathsf{G}\}\subset\mathsf{G}\times\mathsf{G}.$$

The *group manifold* of G is the quotient $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$. The quotient may be identified with G. The group manifold inherits a pseudo-Riemannian metric from the Killing form on $\mathfrak{g}$ which is invariant under right and left multiplication by G. The most classical situation occurs when $\mathsf{G}=\mathsf{PSL}(2,\mathbb{R})$, in which case the group manifold is three-dimensional anti-de Sitter space.

Let Γ be a finitely generated group. Given two representations $\rho,\eta:\Gamma\to\mathsf{G}$, we let $\rho\times\eta:\Gamma\to\mathsf{G}\times\mathsf{G}$ be the product representation defined by $\gamma\mapsto(\rho(\gamma),\eta(\gamma))$. The quotient action of Γ on the group manifold is simply the action given by right multiplication by $\rho(\gamma)$ and left multiplication by $\eta(\gamma)^{-1}$.

7.1. Properness criteria.

Benoist[3] and Kobayashi [44] developed a general criterion to determine when an action on reductive homogeneous spaces are proper. When restricted to our situation we get the following criterion.

Theorem 7.1 (Benoist-Kobayashi properness criterion)**.** *Let* $\rho,\eta:\Gamma\to\mathsf{G}$ *be two representations and assume that* $\rho\times\eta$ *has discrete image and finite kernel. If for every* $K>0$*, the set*

$$\big\{\gamma\in\Gamma:\|\kappa(\rho(\gamma))-\kappa(\eta(\gamma))\|\leq K\big\}$$

is finite, then $\rho\times\eta(\Gamma)$ *acts properly on the group manifold* $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$.

Their properness criterion is most commonly used in the following form.

Corollary 7.2. *Let* $\rho,\eta:\Gamma\to\mathsf{G}$ *be two representations of a finitely generated group and assume that* $\rho\times\eta$ *has discrete image and finite kernel. If there exists* $b>1$ *and* $c\geq0$ *such that*

$$\varpi_1(\kappa(\rho\gamma))-b\varpi_1(\kappa(\eta\gamma))\geq-c,$$

for every $\gamma\in\Gamma$*, then* $\rho\times\eta(\Gamma)$ *acts properly on the group manifold* $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$.

Proof. We consider the functional $\varphi_\mu : \mathfrak{a}\times\mathfrak{a} \to \mathbb{R}$ defined by $\varphi_\mu(x,y) = \varpi_1(x) - b\varpi_1(y)$ and an arbitrary $K > 0$. Since $\mu > 1$ and $c \geq 0$, the set $\{\varphi_\mu \geq -c\} \cap \mathrm{Diag}(\mathfrak{a}^+)$ is compact and thus also is its intersection with the tubular neighborhood

$$\big\{(x,y) \in \mathfrak{a}^+ \times \mathfrak{a}^+ : \|x - y\| \leq K\big\}.$$

Since $\rho \times \eta$ is a proper map (by assumption) and a is also proper, we conclude that the set $\{\gamma \in \Gamma : \|\kappa(\rho(\gamma)) - \kappa(\eta(\gamma))\| \leq K\}$ is finite and thus Benoist-Kobayashi's criterion applies and we may conclude that $\rho \times \eta(\Gamma)$ acts properly. □

7.2. **Ledrappier potentials for Anosov representations.** We now develop some more structure theory for Anosov representations, see Bridgeman-Canary-Labourie-Sambarino [13]. Given a totally Anosov representation $\rho : \Gamma \to \mathsf{G}$ there exists a compact metric space $\mathsf{U}_\rho\Gamma$ equipped with a flow $\phi^\rho = (\phi^\rho_t : \mathsf{U}_\rho\Gamma \to \mathsf{U}_\rho\Gamma)_{t\in\mathbb{R}}$ whose periodic orbits, if Γ is torsion-free, are in one-to-one correspondence with conjugacy classes of infinite order elements of Γ such that the period of $[\gamma]$ is $\varpi_1(\mu(\rho\gamma))$. If Γ has torsion the correspondence is finite-to-one (see Blayac [7] and Carvajales [17, Appendix] for more information).

If $\eta : \Gamma \to \mathsf{G}$ is also totally Anosov, there exists a Hölder-continuous function $f_{\rho\eta} : \mathsf{U}_\rho\Gamma \to \mathbb{R}_+$ such that for every conjugacy class $[\gamma]$ the integral of $f_{\rho\eta}$ over the associated periodic orbit is

$$\int_{[\gamma]} f_{\rho\eta} = \varpi_1(\mu(\eta\gamma)).$$

The function $f_{\rho\eta}$ is called *the Ledrappier potential* of η (seen from ρ) and the map $\eta \mapsto f_{\rho\eta}$, well defined up to Livšic-cohomology, is called the *thermodynamic mapping.* For $\alpha \in (0,1)$, let $\mathrm{H\ddot{o}l}^\alpha(\mathsf{U}_\rho\Gamma,\mathbb{R})$ be the space of Hölder-continuous maps with exponent less than α, equipped with the standard norm that makes it a Banach space.

If $\eta \in \mathfrak{A}_\Delta(\Gamma,\mathsf{G})$ is a smooth point of $\mathrm{Hom}(\pi_1(S),\mathsf{G})$, then there exists a neighborhood $\mathcal{U}_\eta$ of η and $\alpha \in (0,1)$ such that the thermodynamic mapping sends $\mathcal{U}_\eta$ into $\mathrm{H\ddot{o}l}^\alpha(\mathsf{U}_\rho\Gamma,\mathbb{R})$ and it is an analytic map when restricted to $\mathcal{U}_\eta$ (see [13, Proposition 6.2]).

7.3. **Proper affine actions on the group manifold.** We are now ready to establish a weaker version of a recent result due to Ghosh-Kobayashi [35], which suffices for our purposes. We include a proof for completeness.

If U is a smooth open set in $\mathrm{Hom}(\Gamma,\mathsf{G})$, then one can identify $\mathsf{T}U$ with an open set in $\mathcal{C}(\Gamma,\mathsf{G})$ and there exists a smooth map $\Psi : \mathsf{T}U \to \mathrm{Hom}(\Gamma,\mathsf{G})$ with the property that $\frac{d}{dt}\Big|_{t=0}\Psi(\rho,tv) = v$ for all $v \in \mathsf{T}_\rho\mathrm{Hom}(\Gamma,\mathsf{G})$ with $\rho \in U$. We will use the notation (ρ,v) to denote a tangent vector $v \in \mathsf{T}_\rho\mathrm{Hom}(\Gamma,\mathsf{G})$.

Proposition 7.3. *Suppose $\rho_0 \in \mathfrak{A}_\Delta(\Gamma,\mathsf{G})$ is a smooth point of $\mathrm{Hom}(\Gamma,\mathsf{G})$, $v_0 \in \mathsf{T}_{\rho_0}\mathrm{Hom}(\Gamma,\mathsf{G})$ and $\mathsf{u}_{v_0} \in \mathcal{C}(\Gamma,\mathsf{G})$ is the cocycle for $\mathrm{Ad}\,\rho$ associated to v_0. If ρ_0 is semi-simple and $0 \notin \varpi_1\big(\Re(\mathrm{MS}(\rho_0,\mathsf{u}_{v_0}))\big)$, then there exists an open neighborhood V of (ρ_0,v_0) in $\mathsf{T}U$ and $\delta > 0$ so that if $(\rho,v) \in V$ and $t \in (0,\delta)$, the group $\rho \times \Psi(\rho,tv)$ acts properly on $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$.*

Proof. Since $\mathfrak{A}_\Delta(\Gamma,\mathsf{G})$ is open, we may assume that ρ is totally Anosov for all $(\rho,v) \in V$. We may also assume that the set

$$V_0 = \{(\rho,v) \in V \text{ for some } v \in \mathsf{T}_\rho\mathrm{Hom}(\Gamma,\mathsf{G})\}$$

is a smooth open set in $\mathrm{Hom}(\Gamma,\mathsf{G})$. We then consider the Ledrappier potential $f_t(\rho,v) := f_{\rho\Psi(\rho,tv)} : \mathsf{U}_\rho\Gamma \to \mathbb{R}$ from §7.2, with respect to the base representation ρ. Since the map $(\rho,\eta) \mapsto f_{\rho\eta}$ is analytic on an open neighborhood of (ρ,ρ) (which may assume contains $V_0 \times \Psi(V)$), we may find $K > 0$ and $\delta > 0$ such that, after possibly further shrinking V one has

$$\|f_t(\rho,v) - f_0 - t\frac{d}{dt}\Big|_{t=0} f_{\rho\Psi(\rho,tv)}\|_\infty \leq Kt^2$$

for every $(\rho,v) \in V$ and $t \in [0,\delta]$. By integrating the Ledrappier potential over periodic orbits of the geodesic flow, one observes that

$$\big|\varpi_1(\mu(\Psi(\rho,tv)(\gamma))) - \varpi_1(\mu(\rho(\gamma))) - t\mathrm{d}\varpi_1^\gamma(v)\big| \leq Kt^2\varpi_1(\mu(\rho(\gamma)))$$

for every $(\rho, v) \in V$, $t \in [0, \delta]$ and infinite order element $\gamma \in \Gamma$. Since $\varpi_1\big(\Re(\mathrm{MS}(\rho, \mathsf{u}_v))$ varies continuously, we may assume that there exists $c > 0$ so that $\varpi_1\big(\Re(\mathrm{MS}(\rho, \mathsf{u}_v))\big) \leq -c < 0$ for all $(\rho, v) \in V$, so

$$(1 - ct + Kt^2)\varpi_1(\mu(\rho(\gamma))) \geq \varpi_1(\mu(\Psi(\rho, tv)(\gamma))).$$

for every $(\rho, v) \in V$, $t \in [0, \delta]$ and infinite order element $\gamma \in \Gamma$. If we assume that $\delta \leq \frac{c}{2K}$, then

$$\varpi_1(\mu(\rho(\gamma))) \geq b_t \varpi_1(\mu(\Psi(\rho, tv)(\gamma)))$$

for all $t \in (0, \delta]$ and $\gamma \in \Gamma$ with $b_t = \frac{1}{1-ct+Kt^2} > 1$.

Benoist [4, §4.5] showed there exists a finite set $A \subset \Gamma$ so that for all $\gamma \in \Gamma$, there exists $\alpha \in A$ so that $\rho_0(\alpha\gamma)$ is $(\Delta, 2\epsilon)$-proximal. (Recall that $g \subset \mathsf{G}$ is (Δ, δ)-proximal if it is loxodromic, if g^- and g^- are its attracting and repelling flags, then $d(g^+, g^-) > \delta$, $g(B_\delta(g^-)) \subset b_\delta(g^+)$ and g is δ-Lipschitz on $b_\delta(g^+)$ where $B_\delta(g^-)$ is the complement of the open neighborhood of g^- of radius δ and $b_\delta(g^+)$ is the closed neighborhood of g^+ of radius δ.) After possibly shrinking V and δ, one may assume that if $\gamma \in \Gamma$, then there exists $\alpha \in A$ so that if $(\rho, v) \in V$ and $t \in [0, \delta)$, then $\Psi(\rho, tv)(\alpha\gamma)$ is (Δ, ϵ)-proximal. Benoist [4, §4.5] also showed that there exists a compact subset N of $\mathfrak{a}$, depending only on ϵ, so that $g \in \mathsf{G}$ is (Δ, ϵ)-proximal, then $\mu(g) - \kappa(g) \in N$. Therefore, there exists $E > 0$ so that if $g \in \mathsf{G}$ is (Δ, ϵ)-proximal, then

$$\Big|\varpi_1\big(\mu(g)\big) - \varpi_1\big(\kappa(g)\big)\Big| < E.$$

By [5, Cor. 6.34], if $g, h \in \mathsf{G}$, then

$$\varpi_1\big(\kappa(gh)\big)\big) \leq \varpi_1\big(\kappa(g)\big) + \varpi_1(\kappa(h)).$$

Let $S = \max\{\omega_1(\rho_t(\alpha)) \;:\; \alpha \in A \text{ or } \alpha^{-1} \in A,\ t \in [0, \delta]\}$. If $(\rho, V) \in U$, $t \in [0, \delta]$ and $\gamma \in \Gamma$, then there exists $\alpha \in A$ so that $\rho(\alpha\gamma)$ and $\Psi(\rho, tv)(\alpha\gamma)$ are both (Δ, ϵ)-proximal, so

$$\begin{aligned}
\varpi_1(\kappa(\rho(\gamma))) \;&\geq\; \varpi_1(\kappa(\rho(\alpha\gamma))) - S \\
&\geq\; \varpi_1(\mu(\rho(\alpha\gamma))) - (S + E) \\
&\geq\; b_t \varpi_1(\mu(\Psi(\rho, tv)(\alpha\gamma))) - (S + E) \\
&\geq\; b_t \Big(\varpi_1(\kappa(\Psi(\rho, tv)(\gamma))) - (S + E)\Big) - (S + E) \\
&\geq\; b_t \varpi_1(\kappa(\Psi(\rho, tv)(\gamma))) - \mu_t(S + E) - (S + E).
\end{aligned}$$

Since $b_t > 1$ if $t \in (0, \delta)$, Corollary 7.2 implies that $\rho \times \Psi(\rho, tv)\,(\Gamma)$ acts properly on $\mathsf{G} \times \mathsf{G}/\mathrm{Diag}(\mathsf{G})$ for all $(\rho, v) \in V$ and $t \in (0, \delta)$. □

Let $\mathsf{Pr}(S, \mathsf{G}) \subset \mathrm{Hom}(\pi_1(S), \mathsf{G})$ be the set of totally Anosov representations $\rho : \pi_1(S) \to \mathsf{G}$ so that there exists a cocycle u for $\mathrm{Ad}\,\rho$ so that 0 does not lie in $\varpi_1\big(\Re(\mathrm{MS}(\eta, \mathsf{u}))\big)$. Let $\mathsf{Pr}_{sm}(S, \mathsf{G})$ denote the set of semisimple representations $\rho \in \mathsf{Pr}(S, \mathsf{G})$ which are smooth points of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$.

We first observe that $\mathsf{Pr}_{sm}(S, \mathsf{G})$ is non-empty.

Lemma 7.4. *Let G be a complex simple Lie group and $\tilde{\tau} : \mathsf{SL}(2, \mathbb{C}) \to \mathsf{G}$ be the lift of a principal embedding $\tau : \mathsf{PSL}(2, \mathbb{C}) \to \mathrm{Inn}(\mathfrak{g})$. If $\rho : \pi_1(S) \to \mathsf{SL}(2, \mathbb{C})$ is a lift of a quasifuchsian representation which is not fuchsian, then $\tilde{\tau} \circ \rho$ is a smooth point of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$. In particular, $\mathsf{Pr}_{sm}(S, \mathsf{G})$ is non-empty.*

Proof. Goldman [36, §1.2 & 1.3] showed that $\tilde{\tau} \circ \rho$ is a smooth point if the Lie algebra of its centralizer is zero-dimensional. Since lying in the centralizer is an algebraic condition, it suffices to show the Lie algebra of its Zariski closure is zero-dimensional. If $\rho : \pi_1(S) \to \mathsf{PSL}(2, \mathbb{C})$ is quasifuchsian, but not fuchsian, the Lie algebra of the Zariski closure of $\tilde{\tau}\rho(\pi_1(S))$ is $\mathfrak{s} = d\tau(\mathfrak{sl}(2, \mathbb{C}))$.

Suppose that x lies in the centralizer of $\mathfrak{s}$. By Kostant (see [45, Lemma 5.2]), $\mathfrak{s}$ is an $\mathfrak{sl}(2, \mathbb{C})$ triple $\{h, e, f\}$ where h is in the interior of a Weyl chamber with Cartan subalgebra $\mathfrak{h}$ and simple roots Δ such that

$$e = \sum_{\alpha \in \Delta} e_\alpha \qquad e_\alpha \in \mathfrak{g}_\alpha,\ e_\alpha \neq 0.$$

Thus the centralizer of h is $\mathfrak{h}$ and therefore $x \in \mathfrak{h}$. As x is also in the centralizer of e,

$$0 = \mathrm{ad}(x)(e) = \sum_{\alpha\in\Delta} \mathrm{ad}(x)(e_\alpha) = \sum \alpha(x) e_\alpha.$$

It follows that $\alpha(x) = 0$ for all $\alpha \in \Delta$ and as Δ is a basis for $\mathfrak{h}^*$ then $x = 0$. Therefore, $\tilde\tau \circ \rho$ is a smooth point of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$.

Theorem 1.4 implies that there exists a quasifuchsian representation $\sigma : \pi_1(S) \to \mathsf{PSL}(2,\mathbb{C})$ which admit a cocycle u so that $0 \notin \varpi_1\big(\Re(\mathrm{MS}(\rho,\mathsf{u}))\big)$. Let $\rho : \pi_1(S) \to \mathsf{SL}(2,\mathbb{C})$ be a lift of σ, then the proof of Corollary 6.10 implies that $\tilde\tau \circ \rho \in \mathsf{Pr}(S,\mathsf{G})$. The representation $\tilde\tau\circ\rho$ is semi-simple, since its Zariski closure is simple and we have just shown it is a smooth point of $\mathrm{Hom}(\pi_1(S),\mathsf{G})$, so $\tilde\tau\circ\rho \in \mathsf{Pr}_{sm}(S,\mathsf{G})$. □

We now establish the following more precise version of part ii) of Corollary 1.7.

Corollary 7.5. *Suppose that* G *is a complex simple Lie group. The space of representations of* $\pi_1(S)$ *into* $\mathsf{G}\times\mathsf{G}$ *such that the action on* $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$ *is proper has non-empty interior and its closure contains the set* $\{\rho\times\rho : \rho \in \mathsf{Pr}_{sm}(S,\mathsf{G})\}$.

Proof. For each $\rho_0 \in \mathsf{Pr}_{sm}(S,\mathsf{G})$, there exists a cocycle u_0 for $\mathrm{Ad}\,\rho_0$ so that $0 \notin \varpi_1\big(\Re(\mathrm{MS}(\rho_0,\mathsf{u}_0))\big)$. Proposition 7.3 produces $\delta_{\rho_0} > 0$ and a neighborhood V_{ρ_0} of (ρ_0,u_0) in $\mathcal{C}(\Gamma,\mathsf{G})$ so that if $(\sigma,v)\in V_{\rho_0}$ and $t \in (0,\delta_{\rho_0}]$, then $\sigma\times\Psi(\sigma,tv)$ yields a proper action of $\pi_1(S)$ on $\mathsf{G}\times\mathsf{G}/\mathrm{Diag}(\mathsf{G})$. The set

$$\{\sigma\times\Psi(\sigma,tv) : (\sigma,v)\in V_\rho \text{ and } t\in(0,\delta_\rho) \text{ for some } \ \rho\in\mathsf{Pr}_{sm}(S,\mathsf{G})\}$$

is a non-empty open set that verifies the desired conditions. □

8. Angle and bending bounds

In order to show that a Jordan domain is moderately bent, it suffices to have a bound on the angle between geodesics in the intrinsic metric on the boundary of the convex hull and geodesics in $\mathbb{H}^3$ joining points on the boundary. Our final goal is to prove Theorem 1.5 which shows that if $||\beta_\mu||_L < r(L)$, then Ω_μ is moderately bent.

Given a measured lamination μ on $\mathbb{H}^2$ one may construct a locally convex map $f_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ with bending lamination μ. The construction may be formulated in the same language as the construction of a bending deformation. Specifically, for a locally finite lamination μ, we consider the cocycle $Z_\mu : \mathbb{H}^2\times\mathbb{H}^2 \to \mathrm{PSL}(2,\mathbb{C})$. If $x,y\in\mathbb{H}^2$, consider the geodesic arc $[x,y]$ and let $m_1,\dots,m_n$ be the geodesics in the support of μ intersecting $[x,y]$ with atomic measures $a_1,\dots a_m \in \mathbb{R}_+$. Then we define the *bending cocycle*

$$Z_\mu(x,y) = R(m_1, ia_1)R(m_2,ia_2)\dots R(m_{n-1}, ia_{m-1})R(m_n, ia_n).$$

If $x\in m_1$ (or $y \in a_m$), we replace a_1 by $a_1/2$ (similarly a_m by $a_m/2$). Then the map f_μ is defined by fixing a basepoint x_0 and defining

$$f_\mu(x) = Z_\mu(x_0,x)x.$$

This is well-defined up to post-composition by an element of $\mathrm{PSL}(2,\mathbb{C})$. We will denote the image of f_μ by P_μ. This definition can be extended to all measured laminations by taking limits (see [27, Chapter II.3]).

If a locally convex pleated plane P_μ is embedded we call it a *convex pleated plane* and it bounds a convex region X_μ in $\mathbb{H}^3$. If f_μ extends continuously to an embedding $\bar f_\mu : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$, we let Ω_μ denote the Jordan domain of $\partial\mathbb{H}^3$ facing the boundary P_μ of the convex region X_μ. Notice that if $\beta_\pm$ are the bending laminations of $\rho\in\mathrm{QF}(S)$, then the boundary $\partial CH(\rho)$ consists of two convex pleated planes whose bending laminations are the lifts of $\beta_\pm$ to $\mathbb{H}^2$.

8.1. θ-bounded pleated planes.

We now give a concrete definition of the angle we want to bound when P_μ is a convex pleated plane.

Let $\alpha : [0,\infty) \to \mathbb{H}^2$ be a unit speed geodesic ray and $\beta = f_\mu\circ\alpha$. The map β may not be differentiable at every point, but for all t it has a left and right derivative $\beta'_\pm(t)$. If $t>0$, we let $\gamma_t : [0,\infty) \to \mathbb{H}^3$ such that $\gamma_t(0) = \alpha(0)$ and $\gamma_t(r_t) = \beta(t)$ where $r_t = d(\beta(0),\beta(t))$. We then define $\theta^\pm_\mu(t)$ to be the angle at $\beta(t)$ between the tangent vectors $\gamma'_t(r_t)$ and $\beta'_\pm(t)$ at the point $\beta(t)$. If $\beta(t)$ does not lie on an isolated leaf of μ, then $\theta_\alpha^-(t) = \theta_\alpha^+(t)$. Furthermore θ_μ^- is continuous from the left and θ_μ^+ is continuous from the right.

Given $p \neq q \in \mathbb{H}^2$, let $\alpha : \mathbb{R} \to \mathbb{H}^2$ be the unique unit speed geodesic ray with $\alpha(0) = p$ and $\alpha(d(p,q)) = q$. We define

$$\theta_\mu(p,q) = \theta_\mu^+(p,q) = \theta_\alpha^+(d(p,q)) \qquad \theta_\mu^-(p,q) = \theta_\alpha^-(d(p,q)).$$

If q is not on an isolated leaf of μ, then $\theta_\mu^-(p,q) = \theta_\mu^+(p,q) = \theta_\mu(p,q)$.

Given $\theta \in [0,\pi)$ we define μ to be *θ-bounded* if $\theta_\mu(p,q) \le \theta$ for all $p \neq q \in \mathbb{H}^2$. Notice that continuity properties of $\theta_\mu^\pm$ imply that μ is θ-bounded if and only if $\theta_\mu^-(p,q) \le \theta$ for all $p \neq q \in \mathbb{H}^2$.

We first observe that if μ is θ-bounded for some $\theta < \frac{\pi}{2}$, then f_μ is a bilipschitz embedding.

Lemma 8.1. *If $\mu \in \mathcal{ML}(\mathbb{H}^2)$ is θ-bounded for some $\theta < \frac{\pi}{2}$, then $f_\mu : \mathbb{H}^2 \to \mathbb{H}^3$ is a $(\sec\theta)$-bilipschitz embedding. In particular f_μ extends continuously to an embedding $\bar{f}_\mu : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$.*

Proof. By definition f_μ is 1-Lipschitz so we only need to establish the lower bound. We assume first that μ has locally finite support. Let $x, y \in \mathbb{H}^2$ and $\alpha : [0,T] \to \mathbb{H}^2$ be a geodesic parametrized by arc length with $x = \alpha(0)$ and $y = \alpha(T)$. We let $s(t) = d(f_\mu(x), f_\mu(\alpha(t))$ and let t_i be values of t such that $f_\mu(\alpha(t))$ is on a bending line. If $f_\mu \circ \alpha$ does not intersect μ transversely, then it is a geodesic and f_μ is an isometry on α. Otherwise, there is a finite collection $\{t_i\} \subset [0,T]$ so that $f_\mu \circ \alpha$ is geodesic on $[t_i, t_i + 1]$. If $t \neq t_i$ we define $\theta(t) = \theta_\mu(x, \alpha(t))$. Then a simple calculation gives that

$$s'(t) = \cos(\theta(t)) \ge \cos(\theta).$$

It follows that $s(t) \ge \cos(\theta)t$, so

$$d(f_\mu(x), f_\mu(y)) = s(T) \ge \cos(\theta)T = \cos(\theta)d(x,y).$$

For a general θ-bounded lamination μ, we consider the path $\gamma(t) = f_\mu(\alpha(t))$. Then γ is a θ-bounded curve. Let $\mathcal{P}_n$ be an evenly spaced partition of $[0,T]$ with n vertices and let $\gamma_n : [0,T] \to \mathbb{H}^3$ be the piecewise geodesic path with vertices $f_\mu(\alpha(\mathcal{P}_n))$. Then, γ_n is a θ_n bounded curve and $\theta_n \to \theta$, so by taking limits we see that

$$d(f_\mu(x), f_\mu(y)) \ge \cos(\theta)d(x,y).$$

□

We now prove that θ-bounded Jordan domains are moderately bent when $\theta < \frac{\pi}{2}$.

Proposition 8.2. *If $\mu \in \mathcal{ML}(\mathbb{H}^2)$ is θ-bounded for some $\theta < \frac{\pi}{2}$, then Ω_μ is moderately bent.*

Proof. Let (x,y) be a bending pair for Ω_μ. We may assume that $(x,y) = (0,\infty)$ and that the plane Q lying above the real axis $\mathbb{R}$ is a support plane for the pleated plane P_μ. We can assume that Ω_μ contains the upper half-plane. Let P_- and P_+ be the left and right half-planes for the geodesic $\overline{0\infty}$. (Notice that if g is not an atom for the measure, then $Q = P_- = P_+$.)

Suppose that $u \in \partial\Omega_\mu \setminus \{0,\infty\}$ lies to the right of 0. Let $\eta : \mathbb{R} \to \mathbb{H}^3$ be the unique geodesic ray so that $\eta(0) = p = (0,0,|u|) \in \overline{0\infty}$ and $\lim_{t\to+\infty} \eta(t) = u$. Notice that $\eta(\mathbb{R})$ is perpendicular to $\overline{0\infty}$. Let $\gamma : \mathbb{R} \to P_\mu$ be a geodesic in the intrinsic metric on P_μ so that $p = \gamma(0)$ and $\lim_{t\to+\infty}\gamma(t) = u$. Let $\alpha_t : \mathbb{R} \to \mathbb{H}^3$ be a geodesic in $\mathbb{H}^3$ so that $p = \alpha_t(0)$ and $\alpha_t(d(p,\gamma(t))) = \gamma(t)$. Since μ is θ-bounded, the angle $\theta_\mu(\gamma(t),\gamma(0))$ is at most θ. By definition, $\theta_\mu(\gamma(t),\gamma(0))$ is the angle between $-\alpha_t'(0)$ and $-\gamma_-'(0)$, or equivalently, the angle between $\alpha_t'(0)$ and $\gamma_-'(0)$.

Since f_μ extends continuously to an embedding $\bar{f}_\mu : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$, we see that α_t converges to η, so $\alpha_t'(0)$ converges to $\eta'(0)$. Therefore, the angle between $\eta'(0)$ and $\gamma_-'(0)$ is at most θ.

Notice that $\gamma_-'(0)$ lies in the portion of P_- which is to the right of $\overline{0\infty}$. We see, by projecting η onto the plane that the Euclidean line segment R_u joining 0 to u makes an angle of at most θ with portion R_- of ∂P_- which lies to the right of 0. Notice that R_- must be a line segment with positive slope, since otherwise the bending at $\overline{0\infty}$ would be at least $\frac{\pi}{2}$, which would contradict the fact that μ is θ-bounded. Since R_u lies in the lower half plane and makes an angle less than $\frac{\pi}{2}$ with R_- it must have negative slope. Therefore, $\Re(u) > 0$ (see Figure 2).

We may similarly show that if $u \in \partial\Omega_\mu$ lies to the left of 0, then $\Re(u) < 0$. Therefore, if L is the imaginary axis, then L is a round circle in $\hat{\mathbb{C}}$ which intersects $\partial\Omega_\mu$ transversely at the points 0 and ∞ and intersects no other point of $\partial\Omega_\mu$. □

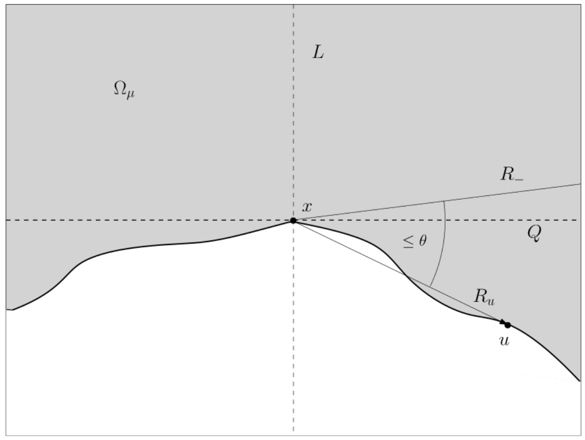

FIGURE 2. Plane L

8.2. **Bending bounds.** We now give explicit bounds on $\|\mu\|_L$ such that the associated pleated plane is θ-bounded and therefore the associated Jordan domain is moderately bounded.

In earlier work [15] the first two authors and Yarmola produced an explicit upper bound on $\|\mu\|_L$ which guarantees that f_μ is a bilipschitz embedding.

Theorem 8.3. (Bridgeman-Canary-Yarmola [15]) *There exists an explicit function $G : \mathbb{R}_+ \to \mathbb{R}_+$ such that if μ is a uniformly bounded measured lamination on $\mathbb{H}^2$ and $\|\mu\|_L < G(L)$, then f_μ is an bilipschitz embedding. In particular, P_μ is a convex pleated plane and f_μ extends continuously to an embedding $\bar{f}_\mu : \partial\mathbb{H}^2 \to \partial\mathbb{H}^3$.*

For all $\theta \in (0, \frac{\pi}{2}]$, we will obtain a bound on $||\mu||_L$ which guarantees that μ is θ-bounded. Our bounds will be defined using the hill function $h : \mathbb{R} \to (0, \pi)$ given by

$$h(t) = \cos^{-1}(\tanh(t)) \quad \text{so} \quad h'(t) = -\text{sech}(t) = -\sin h(t) \quad \text{and} \quad h(0) = \frac{\pi}{2}.$$

This function arises naturally in our situation since if $\alpha : \mathbb{R} \to \mathbb{H}^2$ is geodesic, $x_0 \in \mathbb{H}^2$ does not lie on the geodesic, $s(t) = d(x_0, \alpha(t))$, the angle $\theta(t)$ between the geodesic ray $\overrightarrow{x_0\alpha(t)}$ and $\alpha'(t)$ is monotonically decreasing, then

$$s'(t) = \cos\theta(t) \quad \text{and} \quad \theta'(t) = -\frac{\sin(\theta(t))}{\tanh s(t)} \leq -\sin(\theta(t))$$

(see [28, Lemma 4.4]). We can then define a function $g : \mathbb{R} \to \mathbb{R}$ so that $h(g(t)) = \theta(t)$, so that the pair $(g(t), h(t))$ lies on the graph of the hill function.

Now consider the function

$$u_L(x) = h(x) - Lh'(x)$$

and notice that

$$\lim_{x\to+\infty} u_L(x) = 0 \quad \text{and} \quad \lim_{x\to-\infty} u_L(x) = \pi,$$

since $\lim_{x\to\pm\infty} h'(x) = 0$, $\lim_{x\to+\infty} h(x) = \pi$ and $\lim_{x\to-\infty} h(x) = 0$. One may check that

$$u_L'(x) = -\text{sech}(x) + L\text{sech}(x)\tanh(x) = -\text{sech}(x)(1 - L\tanh(x)),$$

so, if $L \in (0,1]$, then $u_L(x) < 0$ for all $x \in \mathbb{R}$. Thus, if $L \in (0,1]$, u_L is a strictly decreasing function with image $(0,\pi)$. So, given any $\theta \in [0,\pi/2]$ there is a unique value $a_L(\theta)$ so that $u_L(a_L(\theta)) = \theta$ (see Figure 3). We define $r_L(\theta)$ by the equation

$$r_L(\theta) = -Lh'(a_L(\theta)) = L\text{sech}(a_L(\theta)).$$

If $L > 1$, we define

$$r_L(\theta) = -Lh'(L + h^{-1}(\theta)) = L\text{sech}(L + \tanh^{-1}(\cos(\theta)).$$

In all cases, we define $r(L) = r_L(\pi/2)$.

We will give a more explicit description of these functions at the end of this section.

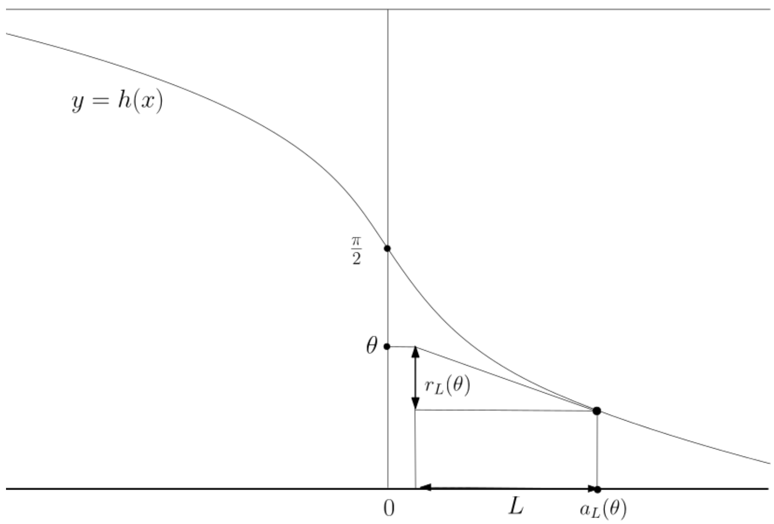

FIGURE 3. Functions $a_L(\theta)$ and $r_L(\theta)$ for $L \leq 1$

We obtain the following strengthening of Theorem 8.3.

Theorem 8.4. *(θ-bounded criterion) Let $\theta \in [0,\pi/2]$ and $L > 0$. If $\mu \in \mathcal{ML}(\mathbb{H}^2)$ and $\|\mu\|_L < r_L(\theta)$, then P_μ is a convex pleated plane and μ is θ-bounded. In particular, if $\|\mu\|_L < r(L)$, then P_μ is θ-bounded for some $\theta < \frac{\pi}{2}$.*

Theorem 1.5 follows immediately from Proposition 8.2 and Theorem 8.4. We will discuss the proof of Corollary 1.6 at the end of the section.

Proof. We first show that, $r_L(\theta) < G(L)$ where G is the function in Theorem 8.3. This guarantees that P_μ is a convex pleated plane. We recall from [15] that

$$G(L) = h(x_0 - L) - h(x_0) = -Lh'(x_0)$$

where x_0 is the unique point such that the tangent line to $(x_0, h(x_0))$ on the graph of h intersects the graph of h again at the point $(x_0 - L, h(x_0 - L))$. They further show that $x_0 \in (0, L)$.

Suppose that $L \leq 1$, Then, by definition,

$$u_L(x_0) = h(x_0) - Lh'(x_0) = h(x_0 - L) > \frac{\pi}{2} \geq \theta = u_L(a_L(\theta)).$$

Since u_L is strictly decreasing, $a_L(\theta) > x_0$. Since h' is increasing on $(0,\infty)$ (but negative),

$$r_L(\theta) = -Lh'(a_L(\theta)) < -Lh'(x_0) = G(L).$$

If $L > 1$, we simply observe that since $x_0 \in (0, L)$ and h' is increasing on $[0,\infty)$, we have

$$r_\theta(L) \leq r_{\pi/2}(L) = -Lh'(L) < -Lh'(x_0) = G(L).$$

We also recall, see [15, Lemma 4.3], that if $\|\mu\|_L < G(L)$, then if $p, q \in \mathbb{H}^2$, then

$$\theta_\mu(p,q) \le h(x_0 - L) < \pi.$$

We first assume that μ is a uniformly bounded measured lamination on $\mathbb{H}^2$ with locally finite support $\mathrm{supp}(\mu)$. Let $\alpha : [0,\infty) \to \mathbb{H}^2$ be a unit-speed geodesic and let $\beta = f_\mu \circ \alpha$. Let $\{t_i\}_{i=1}^r = \alpha^{-1}(\mathrm{supp}(\mu))$ (where $r \in \mathbb{N} \cup \{\infty\}$) ordered so that $t_{i+1} > t_i$ for all i.

For all $t > 0$, we define

$$\theta(t) = \theta^+(t) = \theta_\mu(\alpha(0), \alpha(t)) \quad \text{and let} \quad \theta^-(t) = \theta^-_\mu(\alpha(0), \alpha(t)).$$

Notice that if $t \neq t_i$ (for some i), then $\theta^-(t) = \theta(t)$ and observe that

$$u_i = |\theta^+(t_i) - \theta^-(t_i)| \le \phi_i = \mu(m(\alpha(t_i)))$$

where $\mu(m(\alpha(t_i)))$ is the measure of the leaf of μ passing through $\alpha(t_i)$. One may calculate (see [28, Lemma 4.4]) that if $s(t) = d(\alpha(0), \alpha(t))$ and $t \neq t_i$ for any i, then

$$s'(t) = \cos(\theta(t)) \qquad \theta'(t) = \frac{-\sin(\theta(t))}{\tanh(s(t))} \le -\sin(\theta(t)).$$

Thus on the intervals (t_i, t_{i+1}), the angle $\theta(t)$ is monotonically decreasing from $\theta^+(t_i)$ to $\theta^-(t_{i+1})$.

We define a function $g : (t_1, \infty) \setminus \{t_i\}_{i=1}^r \to \mathbb{R}$ by the property that $h(g(t)) = \theta(t)$. This is well-defined and continuous since h is strictly monotone, continuous and has image $(0, \pi)$. Similarly we define $g^\pm(t_i)$ so that $h(g^\pm(t_i)) = \theta^\pm(t_i)$. This definition guarantees that $H(t) = (g(t), \theta(t))$ and $H^\pm(t_i) = (g^\pm(t_i), \theta^\pm(t_i))$ all lie on the graph of the hill function.

On (t_i, t_{i+1}) the image of $H(t)$ slides downward and to the right. Notice that if $t \in (t_i, t_{i+1})$ and $\theta(t) \in \left(0, \frac{\pi}{2}\right)$, then

$$h'(g(t))g'(t) = \theta'(t) < -\sin(\theta(t)) = -\sin(h(g(t)) = h'(g(t)),$$

so $g'(t) > 1$. In particular, if $\theta(t_i) < \frac{\pi}{2}$, then

$$g^+(t_{i+1}) - g^-(t_i) > t_{i+1} - t_i.$$

At each t_i, the function transitions from $(g^-(t_i), \theta^-(t_i))$ to $(g^+(t_i), \theta^+(t_i))$, so it jumps upward by $u_i \le \phi_i$ and to the left by $g^-(t_i) - g^+(t_i)$.

First, suppose that $L \le 1$ and $\theta \le \frac{\pi}{2}$. Let $a = a_L(\theta)$ and notice that, since $h(a) - Lh'(a) = \theta$, we must have $h(a) < \theta$. Assume that $\theta(t) \ge \theta$ for some t and let

$$T_1 = \inf\{t \in (t_1, \infty) \mid \theta(t) \ge \theta\} \quad \text{and} \quad T_0 = \sup\{t \in (t_1, T_1) \mid \theta^-(t) \le h(a)\}.$$

Notice that since θ^- is continuous from the left, $\theta^-(T_0) \le h(a)$.

Since there is a total upward shift less than $r_L(\theta)$ in the interval $[T_0, T_0 + L)$, we see that

$$\theta(s) < h(a) + r_L(\theta) = h(a) - Lh'(a) = \theta \le \frac{\pi}{2}$$

for all $s \in [T_0, T_0 + L)$. Thus, $T_1 \ge T_0 + L$.

We can make the argument above more concrete in the following fashion. Let $\{t_j, \ldots, t_k\} = \{t_i\} \cap (T_0, T_0 + L)$. If this set is non-empty, let $T_0 = t_{j-1}$ and $T_0 + L = t_{k+1}$, while if the set is empty let $T_0 = t_{j-1}$ (for some arbitrary choice of j) and $T_0 + L = t_j = t_{k+1}$. In either case, $\{t_{j-1}, \ldots, t_{k+1}\}$ is a partition of $[T_0, T_0 + L]$. If $s \in [t_\ell, t_{\ell+1})$ for some $\ell \in \{j-1, \ldots, k\}$, then

$$\begin{aligned}
\theta^-(s) &= \theta^-(T_0) + \left(\sum_{i=j-1}^{\ell} \theta^+(t_i) - \theta^-(t_i)\right) + \left(\sum_{i=j-1}^{\ell-1} \theta^-(t_{i+1}) - \theta^+(t_i)\right) + \theta(s) - \theta(t_\ell) \\
&< h(a) + \left(\sum_{i=j-1} \theta^+(t_i) - \theta^-(t_i)\right) \\
&< h(a) + r_L(\theta).
\end{aligned}$$

Here we use the facts that the third term is strictly negative and the fourth term is non-positive (since $\theta(t)$ is strictly decreasing on each interval) and that the second term is bounded above by $r_\theta(L)$ by assumption.

To complete the proof for $L \le 1$, we now show that $\theta^-(T_0+L) \le h(a)$ which contradicts our definition of T_0. Since $g'(t) > 1$ if $t \ne t_i$ and $\theta(t) \le \frac{\pi}{2}$, there is a horizontal shift right of at least L on the interval $[T_0, T_0+L)$. Since h' is decreasing in $[0,a]$, the total downward shift (associated with the horizontal shift) is greater than $h'(a)L$. Since the total upward shift is less than $r_L(\theta) = -Lh'(a)$, we conclude that $\theta^-(T_0+L) \le h(a)$ giving us our contradiction.

Using the notation above, we can again make this more concrete. Consider the interval $[t_\ell, t_{\ell+1})$ for some $\ell \in \{t_{j-1}, \ldots, t_k\}$. Since $h(g^\pm(t_i)) = \theta^\pm(t_i)$ then by definition of T_0 the interval $[g^+(t_\ell), g^-(t_{\ell+1})]$ is contained in $[0,a]$. Since h' is decreasing on $[0,\infty)$ and $g'(t) > 1$ on $(t_\ell, t_{\ell+1})$, the mean value theorem implies that

$$\theta^-(t_{\ell+1}) - \theta^+(t_\ell) \le h'(a)(g^-(t_{\ell+1}) - g^+(t_\ell)) < h'(a)(t_{\ell+1} - t_\ell).$$

Therefore

$$\begin{aligned}
\theta^-(T_0+L) &= \theta^-(T_0) + \left(\sum_{i=j-1}^{k} \theta^+(t_i) - \theta^-(t_i)\right) + \left(\sum_{i=j-1}^{k} \theta^-(t_{i+1}) - \theta^+(t_i)\right) \\
&< h(a) + \left(\sum_{i=j-1}^{k} \theta^+(t_i) - \theta^-(t_i)\right) + h'(a)\left(\sum_{i=j-1}^{k} t_{i+1} - t_i\right) \\
&\le h(a) + r_L(\theta) + h'(a)L \\
&\le h(a)
\end{aligned}$$

The argument follows the same outline if $L > 1$ and $\theta \le \frac{\pi}{2}$. We define $b = L + h^{-1}(\theta)$, or equivalently, by the equation $h(b-L) = \theta$, so $r_L(\theta) = -Lh'(b)$. As h is decreasing and concave on $[0,\infty)$

$$h(b) < \theta \quad \text{and} \quad h(b) - Lh'(b) < h(b-L) = \theta.$$

Assume that $\theta(t) \ge \theta$ for some t and let

$$T_1 = \inf\{t \in (t_1,\infty) \mid \theta(t) \ge \theta\} \quad \text{and} \quad T_0 = \sup\{t \in (t_1, T_1) \mid \theta^-(t) \le h(b)\}.$$

If $s \in [T_0, T_0+L)$, then

$$\theta(s) < h(b) + r_L(\theta) = h(b) - Lh'(b) < h(b-L) = \theta,$$

so $T_1 \ge T_0 + L$. We argue as before that

$$\liminf_{s \to T_0+L} \theta(s) \le h(b) - Lh'(b) + Lh'(b) = h(b)$$

which again contradicts our definition of T_0 and completes the proof when $L > 1$.

If μ is any measured lamination on $\mathbb{H}^2$ with $\|\mu\|_L < r_L(\theta)$ then by Epstein-Marden-Markovic [29, Lemma 4.6] there exists a sequence μ_n of finite-leaved measured laminations converging to μ such that $\|\mu_n\|_L = \|\mu\|_L$ for all n. By [15, Corollary 4.8] this implies that each map f_{μ_n} is a K-bilipschitz embedding for some K depending only on L and $\|\mu\|_L$. Then by [27, Theorem III.3.11.9] the maps f_{μ_n} converge uniformly on compact sets to f_μ. It follows that as f_{μ_n} are θ-bounded, that f_μ is also θ-bounded. □

Remark: The argument we used for $L > 1$ works for all L. However, if $L \le 1$ and $\theta \le \frac{\pi}{2}$, then $0 < a_L(\theta) < L + h^{-1}(\theta)$, so the argument we give when $L \le 1$, yields a better bound.

8.3. **Explicit description of bounds.** For $L > 1$,
$$r_L(\theta) = -Lh'(L + h^{-1}(\theta)) = L\mathrm{sech}(L + \tanh^{-1}(\cos(\theta)).$$
It follows that
$$r(L) = r_L(\pi/2) = L\mathrm{sech}(L).$$
For $L \leq 1$
$$r_L(\theta) = -Lh'(a_L(\theta)) = L\mathrm{sech}(a_L(\theta))$$
then
$$a_L(\theta) = \cosh^{-1}\left(\frac{L}{r_L(\theta)}\right).$$
As $a_L(\theta)$ is defined by the equation $\theta = u_L(a_L(\theta))$ and
$$u_L(x) = h(x) - Lh'(x) = \cos^{-1}(\tanh(x)) + L\mathrm{sech}(x)$$
then
$$\theta = u_L\left(\cosh^{-1}\left(\frac{L}{r_L(\theta)}\right)\right) = \cos^{-1}\left(\sqrt{1 - \frac{r_L(\theta)^2}{L^2}}\right) + r_L(\theta).$$
Equivalently
$$\theta = \sin^{-1}\left(\frac{r_L(\theta)}{L}\right) + r_L(\theta).$$
Thus r_L is the inverse function of $y = \sin^{-1}(x/L) + x$. Therefore $r_L(\theta)$ is the solution to the equation

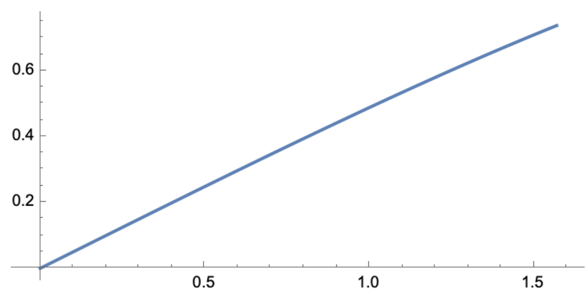

FIGURE 4. Plot of $r_1(\theta)$ on $[0, \frac{\pi}{2}]$

$$x = L\sin(\theta - x).$$
As $r(L) = r_L(\pi/2)$, it follows that r is the inverse function of the function
$$L(x) = x\sec(x).$$
In particular for $L = 1$,
$$r(1) = r_1(\pi/2) \simeq 0.739085.$$

8.4. **Proof of Corollary 1.6.** Theorems 1.1 and 1.5 together imply that if $\rho \in U(S)$ is not fuchsian, then ρ is not a critical point of the entropy function. Theorems 1.2 and 1.5 together imply that if $\rho \in V(S)$ is not fuchsian, then $\mathrm{Ad}\,\rho$ is the linear part of a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$. It remains to show that $US)$ and $V(S)$ are open.

We first observe that if $\alpha : \mathbb{R} \to \mathbb{H}^2$ is a geodesic and μ is a geodesic lamination on $\mathbb{H}^2$, then
$$\mu\left(\alpha([a,b))\right) = \lim_{\epsilon \to 0^+} \mu\left(\alpha((a-\epsilon, b-\epsilon))\right).$$
It follows that $||\mu||_L$ can also be defined to be the supremum of the transverse measure of any open geodesic arc of length L. One can then easily check that $||\cdot||_L$ is a continuous function on $\mathcal{T}(S) \times \mathcal{ML}(S)$. Keen and Series [43] showed that the map from $QF(S)$ to $\mathcal{T}(S) \times \mathcal{ML}(S)$ which takes ρ to $(X^\nu(\rho), \beta_\nu)$ is continuous for both $\nu \in \{\pm\}$. It follows that $U(S)$ and $V(S)$ are open subsets of $QF(S)$. $\square$

9. Teichmüller distance, Schwarzian derivatives and quasicircles

In this section, we obtain versions of our main results in terms of more classical invariants.

Our first description is expressed in terms of the Schwarzian derivative. Given a locally univalent map $f : \Delta \to \hat{\mathbb{C}}$ and $z \in \Delta$, let $M_{f,z}$ be the Möbius transformation with the same 2-jet as f at z. It follows that $M_{f,z}^{-1} \circ f$ has the same two-jet as the identity at z and the *Schwarzian derivative* $S(f)$ of f at z is then given by

$$S(f)(z) = (M_{f,z}^{-1} \circ f)'''(z).$$

In particular, if f is a Möbius transformation, then $S(f) = 0$. In general the Schwarzian derivative measures of how close f is to being Möbius. The Schwarzian derivative is also given by the equation

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

The Schwarzian derivative is a quadratic differential on Δ. Denoting the space of quadratic differentials on a Riemann surface X by $Q(X)$ we define the norm on $Q(X)$ by

$$\|\phi\|_\infty = \sup_{z \in X} \frac{|\phi(z)|}{\rho_X(z)}$$

where $\rho_X(z)|dz|^2$ is the hyperbolic metric on X. Classical results of Nehari [55] show that if $f : \Delta \to \hat{\mathbb{C}}$ is univalent then $\|S(f)\| \le 3/2$ and if $f : \Delta \to \hat{\mathbb{C}}$ is locally univalent and $\|S(f)\| \le 1/2$, then f is univalent.

Let $\rho \in \mathrm{QF}(S)$ with limit set $\Lambda(\rho)$ and complementary Jordan domains $\Omega_+(\rho)$ and $\Omega_-(\rho)$. We let $f_\nu(\rho) : \Delta \to \Omega_\nu$ be the maps uniformizing $\Omega_\nu(\rho), \nu \in \{\pm\}$ and define quadratic differentials $\tilde{\phi}_\nu(\rho) = S(f_\nu(\rho))$ in $Q(\Delta)$. Letting $X_\nu(\rho) = \Omega_\nu/\rho(\pi_1(S))$, then $\tilde{\phi}_\nu(\rho)$ descends to a quadratic differential $\phi_\nu(\rho) \in Q(X_\nu(\rho))$.

By Bers simultaneous uniformization the map $QF(S) \to \mathcal{T}(S) \times \mathcal{T}(\bar{S})$ given by $\rho \to (X_+(\rho), X_-(\rho))$ is a diffeomorphism (see [6]). We let $(X, Y) \to \rho(X, Y)$ be the inverse of this map.

For $X \in \mathcal{T}(S)$ the *Bers embedding* is the map $B_X : \mathcal{T}(\overline{S}) \to Q(X)$ given by

$$B_X(Y) = \phi_+(\rho(X, Y)).$$

Nehari's bounds imply that the image of the Bers embedding B_X in the Banach space $(Q(X), \|\cdot\|_\infty)$ is contained in the ball of radius $3/2$ about 0 and contains the ball of radius $1/2$ about 0. We first show that for a smaller ball about zero, we obtain our criterion.

Theorem 9.1. *If $\rho \in \mathrm{QF}(S)$ and $0 < \|\phi_\nu(\rho)\| < .0739$ for some $\nu \in \{\pm\}$, then ρ is not a critical point of h and there is a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2, \mathbb{C})$ with linear part $\mathrm{Ad}(\rho)$.*

Proof. Let β_ν be the bending laminations of ρ. Bridgeman and Tee [14] show that if $\|\phi_\nu(\rho)\|_\infty < \frac{1}{2}\frac{1}{\sqrt{1+e^{2L}}}$, then

$$\|\beta_\nu\|_L \le 2\tan^{-1}\left(\frac{2\|\phi_\nu(\rho)\|_\infty e^L}{\sqrt{1 - 4\|\phi_\nu(\rho)\|_\infty^2}}\right) = F_L(\|\phi_\nu(\rho)\|_\infty).$$

If we choose $\epsilon > 0$ so that $\epsilon < F_L^{-1}(r(L))$ for some L, then if $\|\phi_\nu(\rho)\|_\infty < \epsilon$ their result implies that

$$\|\beta_\nu\|_L < r(L).$$

Theorems 1.1, 1.2 and 1.5 then imply that ρ is not a critical point of entropy and that there is a proper affine action on $\mathfrak{sl}(2, \mathbb{C})$ with linear part $\mathrm{Ad}\,\rho$.

Recall that $L(r) = r/\cos(r)$ when $r \in (0, r(1)]$ and define $G : (0, r(1)] \to \mathbb{R}$ by $G(r) = F_{L(r)}^{-1}(r)$. So if $r \le r(1)$ and $\epsilon < G(r)$, then $\epsilon < F_{L(r)}^{-1}(r(L))$ and Theorem 9.1 holds for this value of ϵ. Since $r(1) \approx .73908$, we may choose $r = .611$ and compute that

$$G(.611) \approx .0739643$$

so our theorem holds with $\epsilon = .739$. □

We now get a criterion involving the Teichmüller distance between $\Omega_+/\rho(\pi_1(S))$ and $\Omega_0/\rho(\pi_1(S))$.

Theorem 9.2. *If $\rho \in \mathrm{QF}(S)$ and $0 < d_T\left(X_+(\rho), \overline{X_-(\rho)}\right) < .049$, then ρ is not a critical point of h and there is a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$ with linear part equal to $\mathrm{Ad}\,\rho$.*

Proof. Let β_ν be the bending laminations of ρ. Bridgeman and Tee [14] showed that the Bers embedding B_X is $\frac{3}{2}$-Lipschitz with respect to the Teichmüller metric on the domain and L^∞ norm on the image. Thus integrating along the Teichmüller geodesic between $\overline{X_-(\rho)}$ and $X_+(\rho)$ in $\mathcal{T}(S)$ we get

$$\|\phi_-(\rho)\|_\infty \leq \frac{3}{2} d_T\left(X_+(\rho), \overline{X_-(\rho)}\right).$$

Therefore, if $d_T\left(X_+(\rho), \overline{X_-(\rho)}\right) < .049$ then

$$\|\phi_-(\rho)\|_\infty < \frac{3}{2}(.049) = .0735,$$

so our result follows from Theorem 9.1. □

Our next criterion is in terms of the quasiconformal distortion of the limit set. Recall that a Jordan curve is a K-*quasicircle* if it is the image of the unit circle under a K-quasiconformal homeomorphism of $\hat{\mathbb{C}}$. If $\rho \in \mathrm{QF}(S)$, then the limit set $\Lambda(\rho)$ of $\rho(\pi_1(S))$ is a K-quasicircle for some K.

Theorem 9.3. *If $\rho \in \mathrm{QF}(S)$ is not fuchsian and its limit set $\Lambda(\rho)$ is a K-quasicircle for some $K < 1.05$, then ρ is not a critical point of h and there is a proper affine action of $\pi_1(S)$ on $\mathfrak{sl}(2,\mathbb{C})$ with linear part equal to $\mathrm{Ad}\,\rho$.*

In the proof of Theorem 9.3, we will need a version of Theorem 9.2 in the non-equivariant setting, i.e. in terms of the distance between $\Omega_+(\rho)$ and $\Omega_-(\rho)$ in the universal Teichmüller space $\mathcal{T}$.

Roughly, the *universal Teichmüller space*, denoted $\mathcal{T}$, is the space of conformal structures on the unit disk (modulo boundary) which are quasiconformal to the standard conformal structure on the unit disk. Using the measurable Riemann mapping theorem, this can be made explicit using Beltrami differentials as follows (see [50] for further details).

Let $L^\infty(\mathbb{H})$ the space of measurable functions $\mu : \mathbb{H} \to \mathbb{C}$ with essential supremum $\|\mu\|_\infty < \infty$ and $L^\infty_1(\mathbb{H})$ be the open unit ball about zero with respect to the L^∞-norm. If $\mu \in L^\infty_1(\mathbb{H})$, let $\hat{\mu} \in L^\infty_1(\hat{\mathbb{C}})$ be the extension of μ to the lower half-plane by Schwarz reflection, i.e.

$$\hat{\mu}(z) = \begin{cases} \mu(z) & z \in \mathbb{H} \\ \\ \overline{\mu(\overline{z})} & z \in \overline{\mathbb{H}} \end{cases}$$

We then define $w_\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ to be solution to the Beltrami equation $w_{\overline{z}} = \hat{\mu} w_z$ which fixes 0, 1, and ∞. Then $\mathcal{T} = L^\infty_1(\mathbb{H})/\sim$ where $\mu \sim \nu$ if $w_\mu = w_\nu$ on $\mathbb{R}$. We denote by $d_{\mathcal{T}}$ the Teichmüller metric on $\mathcal{T}$ given by

$$d_{\mathcal{T}}([\mu],[\nu]) = \frac{1}{2}\inf_h K(h)$$

where the infimum is over all $h : \mathbb{H} \to \mathbb{H}$ quasiconformal such that $h = w_\nu \circ w_\mu^{-1}$ on $\mathbb{R}$.

Similarly, we let $\overline{\mathcal{T}}$ be universal Teichmüller space of conformal structures quasiconformal to the standard conformal structure on the lower half plane $\overline{\mathbb{H}}$. Schwarz reflection gives an isometry between $(\mathcal{T}, d_{\mathcal{T}})$ and $(\overline{\mathcal{T}}, d_{\overline{\mathcal{T}}})$.

For $[\mu] \in \overline{\mathcal{T}}$ we can extend μ to $\check{\mu} \in L^\infty_1(\hat{\mathbb{C}})$ where $\check{\mu}$ is zero on $\mathbb{H}$ and let w^μ be the solution to the Beltrami equation $w_{\overline{z}} = \check{\mu} w_z$ which fixes $0, 1, \infty$. It follows that f^μ is conformal on $\mathbb{H}$. The Bers embedding $\beta_{\mathbb{H}} : \overline{\mathcal{T}} \to Q(\mathbb{H})$ is then defined by $B_{\mathbb{H}}([\mu]) = S(w^\mu|_{\mathbb{H}})$.

We note that if $\rho \in \mathrm{QF}(S)$ then by definition there exists a quasiconformal map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ conjugating a fuchsian action on $\hat{\mathbb{C}}$ to the quasifuchsian action given by ρ. Therefore f maps $\mathbb{H}$ to $\Omega_+(\rho)$ and $\overline{\mathbb{H}}$ to $\Omega_-(\rho)$ and its Beltrami differential gives a point in $\mathcal{T} \times \overline{\mathcal{T}}$ which we denote by $(\Omega_+(\rho), \Omega_-(\rho))$. The following is the desired generalization of Theorem 9.2.

Theorem 9.4. *If* $\rho \in \mathrm{QF}(S)$ *and* $0 < d_{\mathcal{T}}\left(\Omega_+(\rho), \overline{\Omega_-(\rho)}\right) < .049$*, then* ρ *is not a critical point of* h *and there is a proper affine action of* $\pi_1(S)$ *on* $\mathfrak{sl}(2,\mathbb{C})$ *with linear part equal to* $\mathrm{Ad}(\rho)$*.*

Proof. Bridgeman and Tee [14] also prove that the maps $B_{\mathbb{H}}$ and $B_{\overline{\mathbb{H}}}$ are $\frac{3}{2}$-Lipschitz. Thus integrating along the Teichmüller geodesic between $\overline{\Omega_-(\rho)}$ and $\Omega_+(\rho)$ in $\mathcal{T}$ we get

$$\|\tilde{\phi}_-(\rho)\|_\infty < \frac{3}{2} d_{\mathcal{T}}\left(\Omega_+(\rho), \overline{\Omega_-(\rho)}\right).$$

So if $d_{\mathcal{T}}\left(\Omega_+(\rho), \overline{\Omega_-(\rho)}\right) < .049$, then

$$\|\phi_-(\rho)\|_\infty = \|\tilde{\phi}_-(\rho)\|_\infty < \frac{3}{2}(.049) = .0735$$

Then the result follows from Theorem 9.1. □

We can now prove Theorem 9.3.

Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be the K-quasiconformal homeomorphism mapping $\mathbb{H}$ to $\Omega_+(\rho)$ and $\overline{\mathbb{H}}$ to $\Omega_-(\rho)$. Then

$$d_{\mathcal{T}}(\Omega_+(\rho), \mathbb{H}) \leq \frac{1}{2}\log K(f) \qquad d_{\overline{\mathcal{T}}}(\Omega_-(\rho), \overline{\mathbb{H}}) \leq \frac{1}{2}\log K(f).$$

Therefore

$$d_{\mathcal{T}}(\Omega_+(\rho), \overline{\Omega_-(\rho)}) \leq d_{\mathcal{T}}(\Omega_+(\rho), \mathbb{H}) + d_{\mathcal{T}}(\mathbb{H}, \overline{\Omega_-(\rho)}) \leq \log K(f).$$

Thus if $K < e^{.049} = 1.05022$ then $d_{\mathcal{T}}\left(\Omega_+(\rho), \overline{\Omega_-(\rho)}\right) \leq .049$ and the result follows from Theorem 9.4 above. □

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