EXERCISES FOR THE MINI-COURSE 'ANOSOV REPRESENTATIONS: SOME GENERAL ASPECTS'

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Notation.

-. Let Γ be a hyperbolic group and $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a representation. Consider the associated flat bundle $\mathbb{R}^d \to V_\rho \to \mathsf{U}\Gamma$, defined by $V_\rho = \tilde{\mathsf{U}}\Gamma \times \mathbb{R}^d/\Gamma$, where $\gamma \cdot (p, v) = (\gamma p, \rho(\gamma)v)$. This bundle is equipped with the automorphism $\psi^\rho = (\psi_t^\rho : V_\rho \to V_\rho)_{t\in\mathbb{R}}$, defined as the quotient of

$$(x, v) \mapsto (\tilde{\phi}_t x, v)$$

on $\tilde{\mathsf{U}}\Gamma \times \mathbb{R}^d$, where $\tilde{\phi} = (\tilde{\phi}_t : \tilde{\mathsf{U}}\Gamma \to \tilde{\mathsf{U}}\Gamma)_{t\in\mathbb{R}}$ is the geodesic flow. We will call ψ^ρ the *canonical flat bundle automorphism* of ρ.

-. Given $g \in \mathrm{PGL}(d, \mathbb{R})$ and an inner product in $\mathbb{R}^d$, denote by $\sigma_1(g) \geqslant \cdots \geqslant \sigma_d(g)$ the singular values of g (i.e. the length (in decreasing order) of the ellipse $g\{v : \|v\| = 1\}$). If $\sigma_p(g) > \sigma_{p+1}(g)$ then we say that g has a gap of index p. If this is the case, then $U_p(g)$, the *unstable direction* of g (defined by the vector space spanned by the greatest p axis of the ellipse $g\{v : \|v\| = 1\}$), has dimension p.

Exercise 1. Let $V \to X$ be a vector bundle over a compact base equipped with a bundle automorphism $\psi = (\psi^t : V \to V)_{t\in\mathbb{T}}$ (here $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$).

- *Decay* $\Rightarrow$ *exponential decay.* Assume there exists a ψ-invariant continuous splitting $V = E \oplus F$ such that for every $v \in E$ and $w \in F$ one has

$$\frac{\|\psi^t v\|}{\|\psi^t w\|} \to 0$$

as $t \to +\infty$. Show that $E \oplus F$ is a dominated splitting, i.e. there exist $C, \mu > 0$ such that for all $t \geqslant 0$, $v \in E$ and $w \in F$ one has

$$\frac{\|\psi^t v\|}{\|\psi^t w\|} \leqslant Ce^{-\mu t} \frac{\|v\|}{\|w\|}.$$

- *Coherence of domination.* Assume $E \oplus F$ and $E' \oplus F'$ are dominated splittings for ψ, if $\dim F \leqslant \dim F'$ show that $F \subset F'$ (and $E' \subset E$).

Exercise 2.

- Let $A \in \mathrm{PGL}(d, \mathbb{R})$ have a gap of index p. Show that for all $v \in \mathbb{R}^d$ one has

$$\|Av\| \geqslant \sigma_p(A) \sin \measuredangle(v, U_{d-p}(A^{-1})),$$

conclude that for every subspace Q of dimension $d - p$ one has

$$\|A|_Q\| \geqslant \sigma_p(A)\, d(Q, U_{d-p}(A^{-1})).$$

- Assume $B \in \mathrm{PGL}(d, \mathbb{R})$ is such that AB has a gap of index p, show that

$$d(U_p(A), U_p(AB)) \leqslant \|B\|\,\|B^{-1}\|\, \frac{\sigma_{p+1}(A)}{\sigma_p(A)}.$$

- Using the path shown in Figure 1, show that $\mathbb{Z}^2$ does not admit a dominated representation.

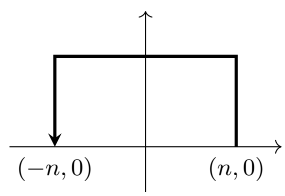

FIGURE 1. A path in $\mathbb{Z}^2$.

Exercise 3.
- Let $\ell \oplus V = \mathbb{R}^d$ be a decomposition of $\mathbb{R}^d$ where $\dim \ell = 1$, show that the tangent space $\mathsf{T}_\ell \mathbb{P}(\mathbb{R}^d)$ can be canonically identified with $\hom(\ell, V)$.
- Show that the map $\mathbb{P}(\mathbb{R}^2) \to \mathbb{R} \cup \{\infty\} = \partial\mathbb{H}^2$ given by $(x, y) \mapsto x/y$ is $\mathrm{PSL}(2,\mathbb{R})$-equivariant.
- Let S be a closed surface of genus $g \geqslant 2$ and let $\rho : \pi_1 S \to \mathrm{PSL}(2,\mathbb{R})$ be induced by the choice of a hyperbolic metric on S. Show that the canonical flat bundle automorphism of ρ has a dominated splitting.

Exercise 4. *Guichard-Wienhard.* Let G be a real rank-1 simple Lie group and consider $\Lambda : G \to \mathrm{PGL}(V)$ a non-trivial representation. Let $\rho : \Gamma \to G$ be an Anosov representation, show that $\Lambda\rho$ is Anosov.

Exercise 5. *Morse Lemma.* Let $(x_n)_{n\in\mathbb{Z}}$ be a bi-infinte (C, μ)-quasi-geodesic on the hyperbolic plane, i.e. for all $n, m \in \mathbb{Z}$ one has

$$\frac{1}{\mu}|n - m| - C \leqslant d_{\mathbb{H}^2}(x_n, x_m) \leqslant \mu|n - m| + C.$$

Use Bochi-Gourmelon's result to show the following weak version of the Morse Lemma: there exists a unique geodesic at bounded Hausdorff distance of $\{x_n : n \in \mathbb{Z}\}$ and the bound only depends on C and μ.

(Hint: It might be convenient to interpret $\mathbb{H}^2$ as the space of inner products on $\mathbb{R}^2$ (up to homothety); bearing this in mind, if $o \in \mathbb{H}^2$ and $g \in \mathrm{PSL}(2,\mathbb{R})$ then:
- Show that
$$d_{\mathbb{H}^2}(o, g \cdot o) = \log \|g\|_o,$$
where $\| \; \|_o$ is the operator norm associated to o,
- Consider the geodesic ray starting at o going through $g \cdot o$, show that its endpoint in $\partial\mathbb{H}^2 = \mathbb{P}(\mathbb{R}^2)$ is the unstable direction $U_1^o(g)$ of g (recall that in general $U_i^o(g)$ is the vector space spanned by the i greatest axis of the ellipse $g \cdot \{v : \|v\|_o = 1\}$).
- Consider $x, y \in \partial\mathbb{H}^2 = \mathbb{P}(\mathbb{R}^2)$ distinct and let σ_{xy} be the geodesic with endpoints x and y, show that the distance
$$d_{\mathbb{H}^2}(o, \sigma_{xy})$$
is coarsely
$$-\log \sin \measuredangle_o(x, y),$$
where $\measuredangle_o(x, y)$ denotes the angle between the lines x and y for the inner product o.)

Exercise 6. *Dominated splitting of $\psi^\rho \Rightarrow \rho$ Anosov.* Let Γ be the fundamental group of a closed negatively curved manifold[1] and $\rho : \Gamma \to \mathrm{PSL}(d, \mathbb{R})$ be a representation.

- Show that if the canonical flat bundle automorphism ψ^ρ admits a dominated splitting $E \oplus F$ with $\dim F = p$ then ρ is p-dominated, i.e. there exist $C, \mu > 0$ such that for all $\gamma \in \Gamma - \{\mathrm{id}\}$ one has
$$\frac{\sigma_{p+1}(\gamma)}{\sigma_p(\gamma)} \leqslant Ce^{-\mu|\gamma|}.$$
Here $|\gamma|$ denotes the word-length of γ w.r.t some generating set and $\sigma_i(g)$ denotes the i-th singular value of the matrix g.
- Show that if $x \in \tilde{\mathrm{U}}\Gamma$ then the vector space $\tilde{E}_x$ only depends on the past of x, i.e. on $\phi_{-\infty}x \in \partial\Gamma$, and that F_x only depends on the future of x, i.e. on $\tilde{\phi}_{+\infty}x \in \partial\Gamma$.
- Conclude that if ψ^ρ admits a dominated splitting then ρ is Anosov.

Exercise 7.

- *Benoist-Guivarc'h's limit set.* Let $\Lambda \subset \mathrm{PGL}(d, \mathbb{R})$ be an irreducible subgroup and assume there exists a proximal g (i.e. there is a g-invariant decomposition $\mathbb{R}^d = g_+ \oplus V_g$, where $\dim g_+ = 1$ and the spectral radius of $g|V_g$ is strictly smaller than the eigenvalue of g_+) in Λ. Show that the set
$$\mathrm{L}_\Lambda = \overline{\{h_+ : h \in \Lambda \text{ proximal}\}}$$
is contained in any closed Λ-invariant subset of $\mathbb{P}(\mathbb{R}^d)$.
- Let $\rho : \Gamma \to \mathrm{PGL}(d, \mathbb{R})$ be a representation equipped with a continuous equivariant map $\xi : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$. Show that the restriction of ρ to the vector space spanned by $\xi(\partial\Gamma)$ is irreducible.

Exercise 8. *Quint's indicator function, basic facts.*[2] Let G be a semi-simple Lie group and $\Lambda \subset G$ be a discrete subgroup. Let $\mathcal{Q}_\Lambda : \mathfrak{a}^+ \to \mathbb{R}_+ \cup \{-\infty\}$ be Quint's growth indicator function, i.e. given a norm $\| \, \|$ on $\mathfrak{a}$ and an open cone $\mathscr{C} \subset \mathfrak{a}^+$ let
$$h_{\mathscr{C}}^{\| \|} = \limsup_{T \to \infty} \frac{1}{T} \log \#\{g \in \Lambda : a(g) \in \mathscr{C} \text{ and } \|a(g)\| \leqslant T\},$$
finally define
$$\mathcal{Q}_\Lambda(v) = \|v\| \inf_{v \in \mathscr{C}} h_{\mathscr{C}}^{\| \|}.$$

- Show that the function $\mathcal{Q}_\Lambda$ does not depend on the chosen norm $\| \, \|$.

Let $\theta : \mathfrak{a}^+ \to \mathbb{R}$ be positively homogeneous (i.e. $\theta(tv) = t\theta(v)$ for all $t \geqslant 0$) and continuous (such as a norm on $\mathfrak{a}$ or an element of $\mathfrak{a}^*$).

- Show that if for all $v \in \mathfrak{a}^+ - \{0\}$ one has $\theta(v) > \mathcal{Q}_\Lambda(v)$ then
$$\sum_{g \in \Lambda} e^{-\theta(a(g))} < \infty.$$

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[1] The statements in this exercise hold true for any hyperbolic group but this would require a technical detour.

[2] Recall that if $(\lambda_n)_{n \in \mathbb{N}}$ is a sequence of positive numbers then the Dirichlet series $L(s) = \sum_{n \in \mathbb{N}} e^{-s\lambda_n}$ has *critical exponent* defined by
$$h = \limsup_{T \to \infty} \frac{1}{T} \log \#\{n \in \mathbb{N} : \lambda_n \leqslant T\};$$
if $s > h$ then $L(s)$ is convergent and if $s < h$ then $L(s) = \infty$.

- Show that if there exists v such that $\theta(v) < \mathcal{Q}(v)$ then
$$\sum_{g\in\Lambda} e^{-\theta(a(g))} = \infty.$$
- Conclude that
$$h^\theta(\Lambda) = \limsup_{T\to\infty} \frac{1}{T}\log \#\{g\in\Lambda : \theta(a(g)) \leqslant T\} = \sup \frac{\mathcal{Q}(v)}{\theta(v)}.$$

Exercise 9. *Growth for linear forms.* Let G be a semi-simple Lie group, $\Lambda \subset G$ be a discrete subgroup and $\mathcal{Q}_\Lambda : \mathfrak{a}^+ \to \mathbb{R}_+ \cup \{-\infty\}$ be Quint's growth indicator function. Consider the set of $\mathfrak{a}^*$ defined by
$$\mathcal{D}_\Lambda = \{\varphi \in \mathfrak{a}^* : \varphi \geqslant \mathcal{Q}\}.$$

- Show that $\mathcal{D}_\Lambda$ is convex.
- Let $\| \ \|$ be a norm on $\mathfrak{a}$ and denote by $\| \ \|^*$ the associated norm on $\mathfrak{a}^*$. Show that
$$h^{\| \ \|}(\Lambda) = \inf_{\varphi\in\mathcal{D}_\Lambda} \|\varphi\|^*.$$