Pressure forms on purely imaginary directions

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Yves's 60th Birthday

$\mathsf{G}_{\mathbb{K}} = \mathsf{PSL}_d(\mathbb{K})$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$,

$\tau =$ inner (Hermitian) product on $\mathbb{K}^d$

The singular values of $g \in \mathsf{PSL}_d(\mathbb{K})$ are the lengths of the axis of the ellipse $g\{v : \|v\| = 1\}$ in decreasing order, denoted by

$$e^{\nu_1(g)} \geq \cdots \geq e^{\nu_d(g)},$$

$\mathsf{E} = \{(a_1, \ldots, a_d) \in \mathbb{R}^d : a_1 + \cdots + a_d = 0\}$

$\nu : \mathsf{G}_{\mathbb{K}} \to \mathsf{E}^+$ is the Cartan projection.

$\sigma_i \in \mathsf{E}^*, \qquad \sigma_i(a_1, \ldots, a_d) = a_i - a_{i+1}.$

$\Pi = \{\sigma_i : i \in \{1, \ldots, d\}\}.$

Anosov representations

Arbitrary rank convex-co-compactness

Introduced by Labourie in '06.

Approach from '14-'15 by Kapovich-Leeb-Porti, Guéritaud-Guichard-Kassel-Wienhard and Bochi-Potrie-S.

$\Gamma =$ discrete group equipped with its word length $||$. $\theta \subset \Pi$.

A "far from walls of elements in θ" quasi-isometric embedding:

$\rho : \Gamma \to \mathsf{G}_{\mathbb{K}}$ is *θ-Anosov* if there exist $\mu > 0, c \geq 0$ such that $\forall \sigma \in \theta$ and all $\gamma \in \Gamma$ one has

$$\sigma\Big(\nu\big(\rho(\gamma)\big)\Big) \geq \mu|\gamma| - c.$$

Open condition in $\hom(\Gamma, \mathsf{G}_{\mathbb{K}})$; stable class of quasi-isometric embeddings.

In '14 Kapovich-Leeb-Porti prove the following[1]:

Theorem (Kapovich-Leeb-Porti)

Let ρ be θ-Anosov, then Γ is Gromov-hyperbolic and there exist continuous equivariant maps

$$\xi_\rho^\theta : \partial\Gamma \to \mathcal{F}_\theta \text{ and } \xi_\rho^{\mathrm{i}\,\theta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\,\theta}$$

such that the product map $(\xi^\theta, \xi^{\mathrm{i}\,\theta})$ sends $\partial^{(2)}\Gamma$ to $\mathcal{F}_\theta^\pitchfork$.

$\mathcal{F}_\theta =$ partial flag of type θ.

$\mathrm{i} : \mathsf{E} \to \mathsf{E}$ is the opposition involution.

$\partial^{(2)}\Gamma =$ pairs of distinct points of $\partial\Gamma$.

$\mathcal{F}_\theta^\pitchfork =$ pairs of transverse partial flags of type θ and $\mathrm{i}\,\theta$.

[1] different proof by Bochi-Potrie-S. in '16

Higher rank phenomenon

Regularity of limit sets does not necessarily decrease after deformation

Yves's work on strictly convex divisible sets $\Rightarrow$ these groups are σ_1-Anosov and have $C^{1+\alpha}$ limit set.

Character variety:

$$\mathfrak{X}(\Gamma, \mathsf{G}_{\mathbb{K}}) = \hom(\Gamma, \mathsf{G}_{\mathbb{K}})/\mathsf{G}_{\mathbb{K}}$$

Anosov characters form an open subspace

$$\mathfrak{X}_{\theta}(\Gamma, \mathsf{G}_{\mathbb{K}}) = \{\rho \in \mathfrak{X}(\Gamma, \mathsf{G}_{\mathbb{K}}) : \rho \text{ is } \theta\text{-Anosov}\}$$

naturally equipped with "geometric structures."

Fix $\rho \in \mathfrak{X}(\Gamma, \mathsf{G}_{\mathbb{K}})$.

$\lambda : \mathsf{G}_{\mathbb{K}} \to \mathsf{E}^{+}$ the Jordan projection.

$\mathcal{L}_{\rho}$ = *Yves's limit cone* = closure of $\mathbb{R}_{+}\lambda\big(\rho(\Gamma)\big)$ and

$$(\mathcal{L}_{\rho})^{*} = \{\varphi \in \mathsf{E}^{*} : \varphi|\mathcal{L}_{\rho} - \{0\} > 0\}.$$

Pressure forms

The moduli space of Anosov representations is canonically equipped with semi-definite symmetric bilinear forms

For $\sigma_p \in \Pi$, denote by $\omega_{\sigma_p} : \mathsf{E} \to \mathbb{R}$ its *fundamental weight:*

$$(a_1, \ldots, a_d) \mapsto a_1 + \cdots + a_p.$$

The *Anosov-Levi* space of ρ :

$$\mathsf{AL}_\rho = \langle \{\omega_\sigma : \rho \text{ is } \sigma\text{-Anosov}\} \rangle \subset \mathsf{E}^*.$$

Bridgeman-Canary-Labourie-S. '13

Every linear form $\varphi \in \mathsf{AL}_\rho \cap (\mathcal{L}_\rho)^*$ defines a positive semi-definite symmetric bilinear form P^φ_ρ on

$$\mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G}_\mathbb{K}).$$

Inspired by Thurston, Bonahon '88, Burger '93, Knieper '95, Bridgeman '08 and McMullen '08.

For $\gamma \in \Gamma$ and $\varphi \in \mathrm{AL}_\rho$ denote by $\varphi^\gamma(\rho) = \varphi\Big(\lambda(\rho(\gamma))\Big)$.

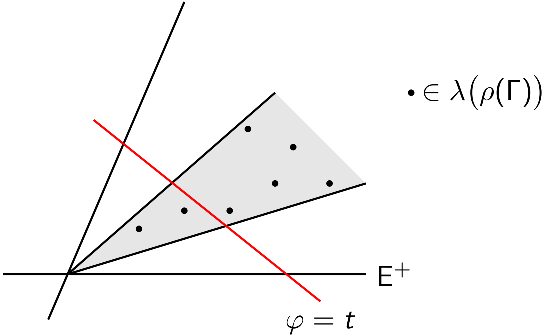
$$R_t^\varphi(\rho) = \{[\gamma] \in [\Gamma] : \varphi^\gamma(\rho) \leq t\}$$

Entropy: exponential growth rate of $t \mapsto R_t^\varphi(\rho)$

$$h^\varphi(\rho) = \lim_{t\to\infty} \frac{\log \# R_t^\varphi(\rho)}{t}$$

Dynamical Intersection: $\mathrm{I}^\varphi(\rho, \eta) = \lim_{t\to\infty} \frac{1}{\# R_t^\varphi(\rho)} \sum_{\gamma \in R_t^\varphi(\rho)} \frac{\varphi^\gamma(\eta)}{\varphi^\gamma(\rho)}$

Both quantities vary analytically.



$$h^\varphi(\rho) = \lim_{t\to\infty} \frac{\log \#R_t^\varphi(\rho)}{t}, \qquad \mathsf{I}^\varphi(\rho,\eta) = \lim_{t\to\infty} \frac{1}{\#R_t^\varphi(\rho)} \sum_{\gamma\in R_t^\varphi(\rho)} \frac{\varphi^\gamma(\eta)}{\varphi^\gamma(\rho)}$$

Proposition (Bridgeman-Canary-Labourie-S. '13)

$$\mathsf{J}^{\varphi}(\rho,\eta)=\frac{h^{\varphi}(\eta)}{h^{\varphi}(\rho)}\mathsf{I}^{\varphi}(\rho,\eta)\geq 1.$$

In particular $\mathrm{Hess}_{\rho}\,\mathsf{J}^{\varphi}(\rho,\cdot)\geq 0$.

Uses heavily the Thermodynamic Formalism of hyperbolic systems developed by Bowen, Ruelle, Ledrappier, Parry, Pollicott among others.

The *pressure form of* φ is $\mathrm{P}^{\varphi}_{\rho} = \mathrm{Hess}_{\rho}\, \mathrm{J}^{\varphi}(\rho, \cdot)$.

Livšic-cohomological independence of ρ and $\dot\rho$?

$\mathrm{P}^{\varphi}_{\rho}(\dot\rho) = 0 \Leftrightarrow \exists K$ such that

$$\forall \gamma \in \Gamma \ \left.\frac{\partial}{\partial t}\right|_{t=0} \varphi^{\gamma}(\rho_t) = K \varphi^{\gamma}(\rho).$$

($K =$ derivative of $\log h^{\varphi}(\rho_t)$)

Questions

Is $\mathrm{P}^{\varphi}_{\rho}$ positive definite?

Does the set $\{d_{\rho}\varphi^{\gamma} : \gamma \in \Gamma\}$ span $\mathrm{T}^{*}_{\rho}\mathfrak{X}(\Gamma, \mathrm{G}_{\mathbb{K}})$?

Does Yves's infinitesimal cone has non-empty interior?

How do different pressure forms relate to each other?

The Hitchin component

and small complex deformations

$S =$ closed connected oriented surface of genus $g \geq 2$.

$\iota_d : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SL}_d(\mathbb{R})$ unique (up to conjugation) d-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{R})$.

A *Fuchsian representation* of $\pi_1 S$ is a representation that factors as

$$\pi_1 S \to \mathrm{PSL}_2(\mathbb{R}) \xrightarrow{\iota_d} \mathrm{PSL}_d(\mathbb{R})$$

where the first arrow is discrete and faithful.

$\mathcal{H}_d(S) =$ The *Hitchin component* of $\mathrm{PSL}_d(\mathbb{R}) =$ the connected component of $\mathfrak{X}\big(\pi_1 S, \mathrm{PSL}_d(\mathbb{R})\big)$ of a(ny) Fuchsian representation.

Theorem (Hitchin '96)

$\mathcal{H}_d(S)$ is diffeomorphic to a real vector space of dimension $(2g-2)\dim \mathrm{PSL}_d(\mathbb{R})$.

Theorem (Labourie '06)

Every Hitchin representation is Π-Anosov and has $\mathrm{C}^{1+\alpha}$ limit curve in projective space $\mathbb{P}(\mathbb{R}^d)$.

Also from Labourie's work (but not directly): for every $\sigma \in \Pi$ the limit set in $\mathcal{F}_\sigma(\mathbb{R}^d)$ is a $\mathrm{C}^{1+\alpha}$ curve.

Study deformations of

$$\pi_S \xrightarrow{\rho} \mathsf{PSL}_d(\mathbb{R}) \to \mathsf{PSL}_d(\mathbb{C})$$

where $\rho \in \mathcal{H}_d(S)$.

Given $\sigma \in \Pi$ let $\mathsf{Hff}_\sigma : \mathfrak{X}_\Pi(\pi_1 S, \mathsf{PSL}_d(\mathbb{K})) \to \mathbb{R}_+$ be

$$\mathsf{Hff}_\sigma(\rho) = \dim_{\mathsf{Hff}}\left(\xi^\sigma_\rho(\partial\pi_1 S)\right)$$

for a metric on $\mathcal{F}_\sigma(\mathbb{K}^d)$ induced by an inner/Hermitian product on $\mathbb{K}^d$.

Non conformality

ρ does <u>not</u> act conformally for such a metric.

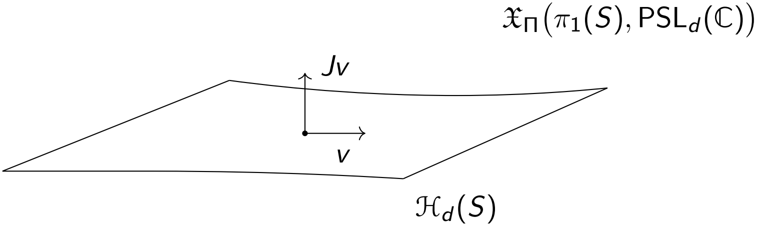
$J^2 = -1$; tensor on $\mathfrak{X}(\pi_1 S, \mathsf{PSL}_d(\mathbb{C}))$ induced by its natural complex structure coming from complex structure of $\mathsf{PSL}_d(\mathbb{C})$.

Theorem (Bridgeman-Pozzetti-S.-Wienhard)

$\rho \in \mathcal{H}_d(S)$ and $v \in \mathsf{T}_\rho \mathcal{H}_d(S)$ then

$$\mathsf{Hess}_\rho \mathsf{Hff}_\sigma(Jv) = \mathsf{P}^\sigma_\rho(v).$$

When $d = 2$ proved by Bridgeman-Taylor '10 and McMullen '08.



$$\mathrm{Hess}_\rho \,\mathrm{Hff}_\sigma(Jv) = \mathrm{P}^\sigma_\rho(v).$$

Theorem (Bridgeman-Canary-Labourie-S. '18)

P^{σ_1} *is a Riemannian metric on* $\mathcal{H}_d(S)$.

Remark: in $\mathcal{H}_{2n}(S)$ the middle root pressure form P^{σ_n} is *degenerate*.

Hints on proof

Key step: Describe Hff_σ as a dynamical quantity; Anosov representations with extra transversality are useful.

Consider $p, q, r \in \{1, \dots, d\}$ with $p + q \leq r$.

Definition

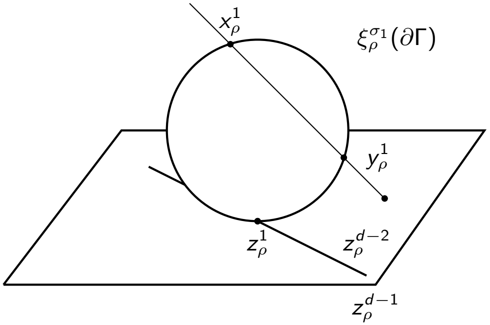
$\rho : \Gamma \to \mathrm{PGL}_d \mathbb{K}$ is (p, q, r)-*hyperconvex* if

- it is $\{\sigma_p, \sigma_q, \sigma_r\}$-Anosov,
- for $x, y, z \in \partial\Gamma$ pairwise distinct one has

$$\left(\xi_\rho^{\sigma_p}(x) \oplus \xi_\rho^{\sigma_q}(y)\right) \cap \xi_\rho^{\sigma_{d-r}}(z) = \{0\}.$$

A $\{\sigma_1\}$-Anosov group preserving a convex set is $(1, 1, d-1)$-hyperconvex.

A $(1,1,2)$-hyperconvex representation of a surface group:



notation: $x_\rho^k = \xi_\rho^{\sigma_k}(x)$.

Theorem (Pozzetti-S.-Wienhard)

$\rho : \Gamma \to \mathrm{PGL}_d(\mathbb{K})$ (p,q,r)-*hyperconvex then*

$$\lim_{y \to x} \measuredangle\left(x_\rho^p \oplus y_\rho^q, x_\rho^r\right) = 0.$$

Theorem (Pozzetti-S.-Wienhard)

$\rho : \Gamma \to \mathrm{PGL}_d(\mathbb{K})$ *(p, q, r)-hyperconvex then* $\lim_{y \to x} \measuredangle\left(x_\rho^p \oplus y_\rho^q, x_\rho^r\right) = 0.$

The convergence property can be used to show that, even though the action of $\rho(\Gamma)$ on $\mathbb{P}(\mathbb{K}^d)$ is not conformal, the action $\rho(\Gamma) \curvearrowright \xi_\rho^{\sigma_1}(\partial\Gamma)$ is.

Theorem (Pozzetti-S.-Wienhard)

Let $\rho : \Gamma \to \mathrm{PSL}_d(\mathbb{K})$ *be* $(1, 1, 2)$*-hyperconvex then* $\mathrm{Hff}_{\sigma_1}(\rho) = h^{\sigma_1}(\rho)$.

Techniques from Bridgeman-Taylor extend once these results have been stablished.

Thanks and Joyeux Anniversaire Yves!!!!