ASYMPTOTIC PROPERTIES OF INFINITESIMAL CHARACTERS AND APPLICATIONS

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ABSTRACT. Inspired by classical results by Benoist, we introduce and study natural objects associated to an integrable tangent vector to the character variety $\mathfrak{X}(\Gamma, \mathsf{G})$ of a semi-group Γ, with values on a semi-simple real algebraic group G of the non-compact type. We obtain non-empty interior of the *cone of Jordan variations* and when G is split we obtain *double-density* results associated to these variations, giving in particular non-empty interior of the set of *length-normalized variations*. We then apply the developed techniques to study pressure forms on the space of Anosov representations, in particular to higher-rank Teichmüller spaces. Among other things we exhibit an *explicit* functional $\varphi \in \mathfrak{a}^*$ whose pressure form is compatible with Goldman's symplectic form at the Fuchsian points of the Hitchin component. We finally exhibit a Diophantine equation that governs the degeneration of the Hausdorff dimension of *higher*-quasi-circles.

CONTENTS

1. Introduction

Let Γ be a semi-group and G a connected semi-simple real-algebraic Lie group of the non-compact type. The *character variety* of Γ with values in G, of morphisms up to conjugation, is denoted by $\mathfrak{X}(\Gamma, \mathsf{G}) = \hom(\Gamma, \mathsf{G})/\mathsf{G}$. In this paper we investigate several objects associated to an integrable tangent vector

$$v \in \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{G}).$$

We will think of v as the (quotient projection of the) derivative of a curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ in $\hom(\Gamma, \mathsf{G})$ with $\rho_0 = \rho$ and such that for every $\gamma \in \Gamma$ the curve $t \mapsto \rho_t(\gamma)$ is real-analytic in some neighborhood of 0.

Let $\mathfrak{a}$ be a Cartan subspace of G, $\mathfrak{a}^+ \subset \mathfrak{a}$ a closed Weyl chamber and denote by $\lambda : \mathsf{G} \to \mathfrak{a}^+$ the Kostant-Jordan-Lyapunov-projection: up to signs, $\exp(\lambda(g))$ is conjugated to the $\mathbb{R}$-diagonalizable element of Jordan's decomposition of g. Commonly, g is *loxodromic* if $\lambda(g) \in \operatorname{int} \mathfrak{a}^+$. For $\gamma \in \Gamma$ we let $\lambda^\gamma : \mathfrak{X}(\Gamma, \mathsf{G}) \to \mathfrak{a}$ be the map $\lambda^\gamma(\eta) = \lambda\big(\eta(\gamma)\big)$ and

$$\mathrm{d}\lambda^\gamma(v) = \frac{\partial}{\partial t}\Big|_{t=0} \lambda(\rho_t(\gamma)) \in \mathfrak{a}$$

its differential at v. For $\varphi \in \mathfrak{a}^*$ we let $\varphi^\gamma : \mathfrak{X}(\Gamma, \mathsf{G}) \to \mathbb{R}$ be the composition

$$\varphi^\gamma = \varphi \circ \lambda^\gamma : \eta \mapsto \varphi\big(\lambda\big(\eta(\gamma)\big)\big).$$

Recall that *Benoist's limit cone* of ρ is defined by $\mathscr{L}_\rho = \overline{\{\mathbb{R}_+ \cdot \lambda^\gamma(\rho) : \gamma \in \Gamma\}}$. Its dual $(\mathscr{L}_\rho)^*$ consists on linear forms which are non-negative on $\mathscr{L}_\rho$, i.e. of $\psi \in \mathfrak{a}^*$ such that $\psi|\mathscr{L}_\rho \geq 0$.

1.1. The cone of Jordan variations.

We introduce the cone

$$\mathscr{V}\mathscr{J}_v := \overline{\Big\{\mathbb{R}_+ \cdot \mathrm{d}\lambda^\gamma(v) : \gamma \in \Gamma \text{ with loxodromic } \rho(\gamma)\Big\}} \subset \mathfrak{a}$$

and call it *the cone of Jordan variations*. For $\psi \in \operatorname{int}(\mathscr{L}_\rho)^*$ we introduce the *set of normalized variations*

$$\mathbb{V}_v^\psi := \overline{\left\{\frac{\mathrm{d}\lambda^\gamma(v)}{\psi^\gamma(\rho)} : \gamma \in \Gamma \text{ with loxodromic } \rho(\gamma)\right\}} \subset \mathfrak{a}.$$

Since we are dealing with semi-simple G we need to rule off variations that occur in proper normal subgroups of G. Let $\mathfrak{g} = \bigoplus_{i\in I} \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ in simple ideals and assume we've chosen the Cartan subspaces $\mathfrak{a}_i$ of $\mathfrak{g}_i$ so that $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$. Let $p_i : \mathfrak{a} \to \mathfrak{a}_i$ be the associated projections. We will say that v has *full loxodromic variation* if for every $i \in I$ one has $p_i(\mathscr{V}\mathscr{J}_v) \neq \{0\}$, so *full* stands for 'non-trivial variation in every simple factor of G' and loxodromic stands for 'the variation is seen on loxodromic elements'. A $\gamma \in \Gamma$ with loxodromic $\rho(\gamma)$ has *full variation* if $\forall i$ $p_i\big(\mathrm{d}\lambda^\gamma(v)\big) \neq 0$.

Let us simplify terminology and say that v *has Zariski-dense base point* if $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$ and $\rho(\Gamma)$ is Zariski-dense in G.

The following statements can be found (respectively) in Corollary 8.4, Proposition 9.1 and Proposition 9.7.

Theorem A. *Let $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{G})$ have Zariski-dense base point and full loxodromic variation. Then, $\mathscr{V}\mathscr{J}_v$ is convex and has non-empty interior. Moreover, full variation elements of $\rho(\Gamma)$ are Zariski-dense in G and their Jordan projections intersect any open subcone of $\mathscr{L}_\rho$. For every $\psi \in \operatorname{int}(\mathscr{L}_\rho)^*$ the set $\mathbb{V}_v^\psi$ is convex.*

The most involved statement is to guarantee non-empty interior of $\mathscr{V}\mathscr{J}_v$. This is of course analogous, and inspired by, the classical result by Benoist [3] stating that if $\rho(\Gamma)$ is Zariski-dense then $\mathscr{L}_\rho$ is convex and has non-empty interior.

Theorem A is stablished by means of the affine geometry $\mathsf{G}\ltimes_{\mathrm{Ad}}\mathfrak{g}$. If $(g,x)\in\mathsf{G}\ltimes\mathfrak{g}$ has loxodromic linear part, i.e. g is loxodromic, then its *Margulis projection* is well defined. This is a conjugacy invariant introduced by Margulis [54, 55] for the affine group $\mathsf{SO}_{2,1}\ltimes\mathbb{R}^3$, when he proved existence of non-abelian free groups acting properly discontinuously on $\mathbb{R}^3$ by affine transformations. In the current context, this projection was defined by Smilga [70].

The bridge between Theorem A and the affine geometry is given by Proposition 8.1 below, independently established Kassel-Smilga [41] and also by Ghosh [27] who further requires that G is split. Recall that a variation $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{G})$ induces a 1-cocycle $\mathsf{u}_v : \Gamma \to \mathfrak{g}$ defined by

$$\mathsf{u}_v(\gamma) = \frac{\partial}{\partial t}\Big|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1}, \tag{1.1}$$

and (ρ,u_v) is a group morphism $\Gamma\to\mathsf{G}\ltimes\mathfrak{g}$. Then, Proposition 8.1 states that the variation of the Jordan projection $\mathrm{d}\lambda^\gamma(v)$ coincides with the $\mathfrak{a}$-coordinate of the Margulis projection of $(\rho(\gamma),\mathsf{u}_v(\gamma))$.

We then study more general affine groups $\mathsf{G}\ltimes_\phi V$ for a class of representations $\phi : \mathsf{G}\to\mathsf{SL}(V)$ and establish *non-empty interior* results in this setting (Corollary 4.6 for irreducible ϕ and Corollary 6.6 when ϕ is reducible but *disjoined*). Similar versions of Corollary 4.6 will also appear in Kassel-Smilga [41] and in Ghosh [29], in particular [41] obtains the convexity stated in Theorem A.

In order to mimic Benoist's proof in [3], we introduce the concept of *Affine Ratio*, an invariant defined for four affine flags in general position; that can also be found in the independent work of Ghosh [27] who deals with split groups (and also proves convexity of spectrum in this case). For example, this invariant vanishes if the four flags concur. We then rely on Smilga's work [72] to relate the defect of additivity of Margulis's invariants to this Affine ratio. These results are achieved in Part 1, however this viewpoint is used in the sequel, specially for Theorem C.

As a consequence we obtain the following. If H is rank 1 and simple, and $\rho:\Gamma\to\mathsf{H}$ is convex co-cocompact, then we let $\hbar(\rho)$ be the Hausdorff dimension of its limit set. Recall from Bridgeman-Canary-Labourie-S. [14] that $\hbar$ is analytic about ρ.

Corollary (Corollary 11.5 - Deformations along level sets of $\hbar$ give non-proper actions)**.** *Let H be the (identity component of the) isometry group of $\mathbb{H}^n_{\mathbb{R}}$ $n\neq 3$, $\mathbb{H}^n_{\mathbb{H}}$ $n\geq 2$, or the Cayley hyperbolic plane. Let $v\in\mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{H})$ have Zariski-dense and convex-co-compact base-point, if $\mathrm{d}\hbar(v)=0$ then the action (ρ,u_v) on $\mathfrak{h}$ is not proper. In particular, if $\hbar$ is critical at ρ then there is no proper affine action on $\mathfrak{h}$ above* $\mathrm{Ad}\,\rho$.

In the case of $\mathbb{H}^2_{\mathbb{R}}$, one should recall the work of Mess [57], Goldman-Labourie-Margulis [31] and the alternative proof given by Danciger-Guéritaud-Kassel [21].

1.2. **Base point Livšic-independence.** Denote by Δ the set of simple restricted roots associated to $\mathfrak{a}^+$. For $\sigma\in\Delta$ let $\mathfrak{g}_\sigma$ be its root-space (Eq. (2.1)). For a non-empty $\vartheta\subset\Delta$ the subspace[1] $\mathfrak{a}_\vartheta=\bigcap_{\sigma\in\Delta-\vartheta}\ker\sigma$ comes equipped with a natural projection $\pi_\vartheta:\mathfrak{a}\to\mathfrak{a}_\vartheta$ (see § 2.6). We let

$$\lambda_\vartheta=\pi_\vartheta\circ\lambda,$$
$$\mathbb{V}^\psi_{\vartheta,v}=\pi_\vartheta\big(\mathbb{V}^\psi_v\big).$$

In Corollary 10.5 we prove the following:

Theorem B (Double-density for roots with multiplicity 1)**.** *Let $\rho:\Gamma\hookrightarrow\mathsf{G}$ be a Zariski-dense sub-semi-group and consider an integrable, full loxodromic variation $v\in\mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{G})$. Let $\vartheta\subset\Delta$ be such that $\dim\mathfrak{g}_\sigma=1$ for all $\sigma\in\vartheta$, then the additive group spanned by*

$$\big\{\big(\mathrm{d}\lambda^\gamma_\vartheta(v),\lambda^\gamma(\rho)\big):\gamma\in\Gamma\textit{ with loxodromic }\rho(\gamma)\big\}$$

is dense in $\mathfrak{a}_\vartheta\times\mathfrak{a}$. In particular, for any $\psi\in\operatorname{int}(\mathcal{L}_\rho)^*$ the convex set $\mathbb{V}^\psi_{\vartheta,v}$ has non-empty interior.

Before passing to the next subsection we show two applications of Theorem B. The first one can be found in Corollary 11.7, we refer the reader to §2.13 for the definition of Anosov representations, introduced by Labourie [48] and generalized by Guichard-Wienhard [35]. Recall also that for $\rho\in\mathfrak{X}(\Gamma,\mathsf{SL}(3,\mathbb{R}))$ the *Hilbert entropy* is defined by

$$\hbar^{\mathsf{H}}_\rho=\lim_{t\to\infty}\frac{1}{t}\log\#\Big\{[\gamma]\in[\Gamma]:\frac{(\lambda^\gamma_1-\lambda^\gamma_3)(\rho)}{2}\leq t\Big\}.$$

Corollary (No proper actions above level sets of entropy)**.** *Consider a Δ-Anosov $\rho:\Gamma\to\mathsf{SL}(3,\mathbb{R})$ with Zariski-dense image and $0\neq v\in\mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{SL}(3,\mathbb{R}))$. If $\mathrm{d}\hbar^{\mathsf{H}}(v)=0$ then the affine action on $\mathfrak{sl}(3,\mathbb{R})$ via u_v is not proper. Moreover, there is a neighborhood $\mathcal{U}$ of (ρ,v) in $\mathsf{T}\mathfrak{X}(\Gamma,\mathsf{SL}(3,\mathbb{R}))$ such that for all $(\eta,w)\in\mathcal{U}$ the action via u_w is also not proper.*

The second one can be found in Corollary 13.6, the definition of the Hitchin component $\mathcal{H}_\mathfrak{g}(S)$ of S associated to a simple split $\mathfrak{g}$ can be found in §1.3.

Corollary (Curves with arbitrary small root-variation)**.** *Let $\mathfrak{g}$ be simple split, $\sigma\in\Delta$, $\varpi_\sigma\in\mathfrak{a}^*$ be the associated fundamental weight and $0\neq v\in\mathsf{T}_\rho\mathcal{H}_\mathfrak{g}(S)$ have Zariski-dense base-point. Then, there exists $h>0$ such that for positive ε and δ there exists $C>0$ with*

$$\#\big\{[\gamma]\in[\pi_1S]\textit{ primitive}:\varpi^\gamma_\sigma(\rho)\in(t-\varepsilon,t]\textit{ and }|\mathrm{d}\sigma^\gamma(v)|\leq\delta\big\}\sim C\frac{e^{ht}}{t^{3/2}}.$$

In particular, for every $\delta>0$ there exists $\gamma\in\pi_1S$ with arbitrary large translation length and such that $|\mathrm{d}\sigma^\gamma(v)|\leq\delta$.

[1]Conventions are made so that $\mathfrak{a}_\Delta=\mathfrak{a}$.

1.3. **Pressure forms for higher-rank Teichmüller spaces.** A fundamental question in higher rank Teichmüller theory consists on finding an analog of the Weil-Petterson Kähler metric for the Hitchin component.

Let now $\mathfrak{g}$ be a simple split Lie algebra and $\operatorname{Inn}\mathfrak{g}$ be its group of inner automorphisms. Recall from Kostant [46] that $\mathfrak{g}$ contains a remarkable $\operatorname{Inn}\mathfrak{g}$-conjugacy class of $\mathfrak{sl}_2(\mathbb{R})$ embeddings called *the principal* $\mathfrak{sl}_2$*'s*. The *Hitchin component of* $\mathfrak{g}$ (or of $\operatorname{Inn}\mathfrak{g}$, or of the type of $\mathfrak{g}$) of a closed connected orientable surface S with genus ≥ 2, is a(ny) connected component of the character variety

$$\mathcal{H}_{\mathfrak{g}}(S) \subset \mathfrak{X}(\pi_1 S, \operatorname{Inn}\mathfrak{g})$$

characterized by the following fact: there exists a discrete and faithful $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ whose Zariski-closure is a principal $\mathsf{PSL}(2,\mathbb{R})$ in $\operatorname{Inn}\mathfrak{g}$. The latter representations are called *Fuchsian* and the space of Fuchsian representations forms a natural embedding of the Teichmüller space $\mathcal{T}(S) = \mathcal{H}_{\mathsf{A}_1}(S)$ of S inside $\mathcal{H}_{\mathfrak{g}}(S)$.

Hitchin [37] showed that $\mathcal{H}_{\mathfrak{g}}(S)$ is a contractible analytic manifold and Labourie [48] - Beyrer-Guichard-Labourie-Pozzetti-Wienhard [6] show that every $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ is faithful with discrete image (see also Fock-Goncharov [25]).

The space $\mathcal{H}_{\mathfrak{g}}(S)$, being a subset of a surface-group character variety, is naturally equipped with Goldman's [30] symplectic form ω. Moreover, Bridgeman-Canary-Labourie-S. [14] construct, for each $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ and each linear form $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$, a semi-definite symmetric bilinear form $\mathbf{P}^{\psi}_{\rho}$ on $\mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)$, called the ψ*-pressure form* (see § 2.12 for references on similar constructions).

The question from the beginning of this section can be interpreted as a compatibility question between the pressure forms $\mathbf{P}^{\psi}$, for different choices of ψ, and ω. Combining Labourie-Wentworth [52] with Corollary 8.6 and §14 we establish:

Corollary (Corollary 16.4)**.** *We let* $\mathfrak{g}$ *have type* A*,* B*,* C *or* G_2*. Then there exist a unique (up to scaling) and explicit form* $\varphi \in \mathfrak{a}^*$ *such that* $\mathbf{P}^{\varphi}$ *is compatible with Goldman's symplectic form on* $\mathcal{H}_{\mathfrak{g}}(S)$ *at the Fuchsian points.*

The form φ is explicit but rather involved to compute. For the rank 2 simple split Lie algebras one has (see Remark 16.5), up to scaling:

$$\begin{aligned}\varphi_{\mathfrak{sl}(3,\mathbb{R})}(a) &= a_1 - a_3 - \frac{\sqrt{2}}{2}a_2;\\ \varphi_{\mathfrak{sp}(4,\mathbb{R})}(a) &= \big(3+\frac{\sqrt{10}}{30}\big)a_1 + \big(1-\frac{\sqrt{10}}{10}\big)a_2;\\ \varphi_{\mathfrak{G}_2}(a) &= \Big(8+\frac{\sqrt{42}}{315}\Big)a_1 + \Big(2-\frac{\sqrt{42}}{210}\Big)a_2. \end{aligned} \tag{1.2}$$

Observe that in all these cases $\varphi \in (\mathfrak{a}^+)^*$ so Theorem C below implies that $\mathbf{P}^{\varphi}$ is Riemannian on the corresponding Hitchin component. However, uniqueness of φ suggests the need to work with the following concept. We let $\operatorname{Mod}(S) = \operatorname{Out}(\pi_1 S)$ be the group of outer automorphisms of $\pi_1 S$, it acts by precomposition on the character variety of $\pi_1 S$.

Definition 1.1. A *length function on* $\mathcal{H}_{\mathfrak{g}}(S)$ is a smooth $\operatorname{Mod}(S)$-invariant map $\psi : \mathcal{H}_{\mathfrak{g}}(S) \to \mathfrak{a}^*$ such that for all ρ one has $\psi(\rho) \in \operatorname{int}(\mathcal{L}_\rho)^*$.

Indeed, it follows from Kim-Zhang [43] and Labourie [50] that rank 2 Hitchin components carry a 1-parameter family of $\operatorname{Mod}(S)$-invariant Kähler metrics, so its seems natural to expect that the length functional $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$ so that $\mathbf{P}^\psi$ is compatible with ω at ρ, (if it exists) depends on ρ.

The pressure forms $\mathbf{P}^\psi$ only depend on $\psi \in \mathfrak{a}^*$ up to scaling, so non-constant length functions are purely a higher-rank phenomenon. A natural choice is, for example, Quint's growth form: we fix a norm N on $\mathfrak{a}$ and let $\psi(\rho) =$ the unique ψ in the critical hyper-surface $\mathcal{Q}_{\rho,\Delta}$ of ρ minimizing the dual norm N^*, (see Eq. (2.15)).

In this paper we prove the following (see Corollary 12.10 for the corresponding statement for Θ-positive representations into $\mathsf{SO}(p,q)$).

Corollary (Corollary 12.14)**.** *For any length function $\psi : \mathcal{H}_{\mathfrak{g}}(S) \to \mathfrak{a}^*$ the associated pressure semi-norm $\rho \mapsto \mathbf{P}^{\psi(\rho)}$ induces a $\operatorname{Mod}(S)$-invariant path metric on $\mathcal{H}_{\mathfrak{g}}(S)$. If moreover $\mathfrak{g}$ has type A, B, C, D, or G_2 and ψ is chosen as to not verify any of the degenerations in Theorem C, then $\rho \mapsto \mathbf{P}^{\psi(\rho)}$ is a $\operatorname{Mod}(S)$-invariant Riemannian metric on $\mathcal{H}_{\mathfrak{g}}(S)$, as regular as ψ.*

The proof essentially boils down to understanding the degenerating set of $\mathbf{P}^\psi$ for a fixed $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$. This is the content of Theorem C below which we now explain.

Let us chose a principal $\mathfrak{sl}_2$, $\mathfrak{s}$, whose semi-simple element lies in $\mathfrak{a}$ (and with Weyl chamber contained in $\mathfrak{a}^+$). Recall from Kostant [46] that the decomposition of $\mathfrak{g}$ into irreducible $\operatorname{ad}\mathfrak{s}$-factor has rank $\mathfrak{g}$ factors, each of them of odd dimension $2e+1$. The numbers e appearing in this decomposition are called *the exponents* of $\mathfrak{g}$ and we denote by V_e the associated irreducible factor, so that

$$\mathfrak{g} = \bigoplus_{e \text{ exponent of } \mathfrak{g}} V_e$$

is the decomposition of $\mathfrak{g}$ into irreducible $\operatorname{ad}\mathfrak{s}$-factors. Table 2 in §14 gives the exponents for each type of $\mathfrak{g}$.

Definition 1.2. If e is an exponent of $\mathfrak{g}$ then we consider the line $\varkappa^e = V_e \cap \mathfrak{a}$ and call it *the e-th Kostant line.*

The family $\{\varkappa^e : e \text{ exponent of } \mathfrak{g}\}$ spans $\mathfrak{a}$.

Identifying the tangent space at ρ to the character variety with the first twisted cohomology group $H^1_{\operatorname{Ad}\rho}(\pi_1 S, \mathfrak{g})$ as in Eq. (1.1), the above decomposition of $\mathfrak{g}$ in $\mathfrak{s}$-modules yields a splitting at a Fuchsian point $\delta \in \mathcal{H}_{\mathfrak{g}}(S)$,

$$\mathsf{T}_\delta \mathcal{H}_{\mathfrak{g}}(S) = \bigoplus_{e \text{ exponent of } \mathfrak{g}} H^1_{\operatorname{Ad}\delta}(\pi_1 S, V_e) = \bigoplus_{e \text{ exponent of } \mathfrak{g}} \mathsf{T}^e_\delta,$$

where we have simplified notation and writen $\mathsf{T}^e_\delta = H^1_{\operatorname{Ad}\delta}(\pi_1 S, V_e)$.

Let us denote by $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ the opposition involution, if non-trivial, it is realized by an external involution of $\operatorname{Inn}\mathfrak{g}$ that induces an involution $\underline{\mathrm{i}} : \mathfrak{X}(\pi_1 S, \operatorname{Inn}\mathfrak{g}) \to \mathfrak{X}(\pi_1 S, \operatorname{Inn}\mathfrak{g})$. Points in $\mathcal{H}_{\mathfrak{g}}(S)$ that are fixed by $\underline{\mathrm{i}}$ will be called *self-dual.* If ρ is self-dual then $\mathrm{d}_\rho \underline{\mathrm{i}}$ is an involution on $\mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)$.

In the special case of D_4, its Dynkin diagram has an order three automorphism τ that induces an order three automorphism $\underline{\tau}$ of $\mathcal{H}_{\mathsf{D}_4}(S)$. Also, in this case 3 appears twice as an exponent (see Table 2), let us denote by $V_{3,\mathrm{a}}$ the $\operatorname{ad}\mathfrak{s}$-factor that is not contained in the natural representation $\mathfrak{so}(3,4) \to \mathfrak{so}(4,4)$ preserving a non-isotropic line, see § 14.3.1 for details.

With these notations at hand we can completely describe the degenerations of pressure forms on the Hitchin component of classical type.

Theorem C (Pressure degenerations are Lie-theoretic)**.** *Let $\mathfrak{g}$ be simple split of type A, B, C, D or G_2. Consider $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ and $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$ then, the pressure form $\mathbf{P}^{\psi}_{\rho}$ is degenerate at $v \in \mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)$ if and only if either of the following hold:*

- *ρ is Fuchsian and*
$$v \in \bigoplus_{e:\psi(\varkappa^e)=0} \mathsf{T}^e_\rho,$$
- *ρ is self dual, ψ is* i*-invariant and v is $\mathrm{d}_\rho\bar{\mathrm{i}}$-anti-invariant.*
- *$\mathfrak{g}$ is of type A_6 or C_3, the Zariski closure of $\rho(\pi_1 S)$ is conjugate to the 7-dimensional irreducible representation of the real split group G_2, $\psi(\varkappa^3) = 0$ and $v \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3)$.*
- *$\mathfrak{g}$ is of type D_4, the Zariski closure of $\rho(\pi_1 S)$ has Lie algebra conjugate to the spin representation $\mathfrak{so}(3,4) \to \mathfrak{so}(4,4)$, $v \in H^1_{\mathrm{Ad}\,\rho}\big(\pi_1 S, \underline{\tau}(V_{3,\mathrm{a}})\big)$ and $\psi(-1,1,1,-1) = 0$.*

For example, in $\mathsf{PSL}(4,\mathbb{R})$ the Kostant lines are

$$\begin{aligned}\varkappa^1 &= \mathbb{R}\cdot(3,1,-1,-3),\\ \varkappa^2 &= \mathbb{R}\cdot(1,-1,-1,1),\\ \varkappa^3 &= \mathbb{R}\cdot(1,-3,3,-1),\end{aligned}$$

so the strongly dominant weight $2\varpi_1 + \varpi_2 : a \mapsto 3a_1 + a_2$ contains $\varkappa^3$ in its kernel. Consequently, Theorem C implies that the Pressure form $\mathbf{P}^{2\varpi_1+\varpi_2}$ degenerates *only* at the Fuchsian locus and in the directions given by T^3_ρ. See Table 3 in §14 for the list of Kostant lines on $\mathfrak{sl}(d,\mathbb{R})$ for $d \leq 8$.

Remark 1.3. The question of non-degeneration on the Hitchin component has been dealt with in the previously mentioned work by Bridgeman-Canary-Labourie-S. [14], where it is stablished that:

- if we let ϖ_1 be the first fundamental weight $\varpi_1 : a \mapsto a_1$, then $\mathbf{P}^{\varpi_1}$ is Riemannian on $\mathcal{H}_{\mathfrak{sl}(d,\mathbb{R})}(S)$ and,
- for every strongly dominant weight χ the form $\mathbf{P}^{\chi}$ is Riemannian on the space $\{\rho \in \mathcal{H}_{\mathfrak{g}}(S) : \rho(\pi_1 S) \text{ is Zariski-dense}\}$;

and in B.-C.-L.-S. [15] where it is stablished that $\mathbf{P}^{\sigma_1}$ is Riemannian, where $\sigma_1 : a \mapsto a_1 - a_2$ is the first simple root. In [14] it is also established the following (to be compared with Theorem B): *Let $\rho_t : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ be an analytic curve of irreducible representations with speed v and assume there exist $g, h \in \Gamma$ such that $\rho_0(g)$ and $\rho_0(h)$ are bi-proximal and transverse (see Def. 2.7), assume also that $\mathrm{d}\varpi_1^g(v) \neq 0$, then the set of pairs*

$$\big\{\big(\mathrm{d}\varpi_1^\gamma(v), \varpi_1^\gamma(\rho)\big) : \gamma \in \Gamma \textit{ with } \rho_0(\gamma) \textit{ proximal}\big\} \subset \mathbb{R}^2$$

is not contained in a line.

Finally, pressure forms for ϖ_1 and σ_1 have been shown to be Riemannian on the Hitchin components of geometrically finite Fuchsian groups by Bray-Canary-Kao-Martone [12].

1.4. **Hausdorff dimension of higher-quasi-circles.** We then use Theorem C to investigate deformations of higher-rank Teichmüller spaces inside the complexified ambient group.

Let us fix $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$. Then it follows from Labourie [48] that there exists a ρ-equivariant Hölder-continuous map

$$\zeta_\rho : \partial\pi_1 S \to \mathcal{F}_\Delta(\operatorname{Inn}\mathfrak{g}) \tag{1.3}$$

from the Gromov-boundary of $\pi_1 S$ to the full flag space of $\operatorname{Inn}\mathfrak{g}$. For type A, this curve should be interpreted as a Hölder $\pi_1 S$-equivariant analogue, of the analytic $\mathsf{PSL}(2,\mathbb{R})$-equivariant Veronesse map $\mathbb{P}^1 \to \mathbb{P}^{d-1}$. While the circle $\zeta(\partial\pi_1 S) \subset \mathcal{F}_\Delta(\operatorname{Inn}\mathfrak{g})$ is only Lipschitz, each of its projections into the maximal flags

$$\mathbf{L}_{\rho,\sigma} := \zeta_\rho^\sigma(\partial\pi_1 S) \subset \mathcal{F}_\sigma(\operatorname{Inn}\mathfrak{g}),$$

for $\sigma \in \Delta$, is a $\mathrm{C}^{1+\alpha}$ circle (Labourie [48] and Pozzetti-S.-Wienhard [62]). In this paper we deform these circles inside the complex maximal flag variety $\mathcal{F}_\sigma(\operatorname{Inn}(\mathfrak{g}_\mathbb{C}))$. More precisely, the equivariant map from Equation (1.3) can also be defined for representations neighboring ρ in the complex characters $\mathfrak{X}\big(\pi_1 S, \operatorname{Inn}(\mathfrak{g}_\mathbb{C})\big)$ ([48]). Moreover, there exists a neighborhood $\mathcal{U}$ of $\mathcal{H}_{\mathfrak{g}}(S) \subset \mathfrak{X}\big(\pi_1, \operatorname{Inn}(\mathfrak{g}_\mathbb{C})\big)$ such that for every $\eta \in \mathcal{U}$ and every $\sigma \in \Delta$ the function

$$\begin{aligned} \operatorname{Hff}_\sigma : \mathcal{U} &\to [1,\infty) \\ \eta &\mapsto \operatorname{Hff}(\mathbf{L}_{\eta,\sigma}) \end{aligned}$$

is real-analytic[2], where Hff denotes the Hausdorff dimension for a Riemannian metric on $\mathcal{F}_\sigma(\operatorname{Inn}(\mathfrak{g}_\mathbb{C}))$. We can thus study its Hessian at the critical points $\mathcal{H}_{\mathfrak{g}}(S)$.

If we let J be the tensor squaring $-\operatorname{id}$ on the complex characters $\mathfrak{X}(\pi_1 S, \operatorname{Inn}(\mathfrak{g}_\mathbb{C}))$, induced by the complex structure of $\operatorname{Inn}(\mathfrak{g}_\mathbb{C})$, then the tangent space at a Hitchin point $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ naturally splits as

$$\mathsf{T}_\rho\mathfrak{X}\big(\pi_1 S, \operatorname{Inn}(\mathfrak{g}_\mathbb{C})\big) = \mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S) \oplus \mathsf{J}\Big(\mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)\Big).$$

Deformations along $\mathsf{T}\mathcal{H}_{\mathfrak{g}}(S)$ are understood, $\operatorname{Hff}_\sigma \equiv 1$, so we turn into the complementary factor. By means of Bridgeman-Pozzetti-S.-Wienhard [16] and Theorem C, in § 17 we establish the following (an analogous result for Θ-positive representations can be found on Corollary 12.7):

Corollary (Corollary 17.2)**.** *Let $v \in \mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)$ be non-zero and have Zariski-dense base point, then*

$$\operatorname{Hess}_\rho \operatorname{Hff}_\sigma(\mathsf{J}v) > 0.$$

In particular there exists a neighborhood (in the complex characters) of points in $\mathcal{H}_{\mathfrak{g}}(S)$ with Zariski-dense image where $\operatorname{Hff}_\sigma$ is rigid, i.e. such that if $\operatorname{Hff}_\sigma(\eta) = 1$ then η has values in the real characters.

The second statement is a local analog of a classical result of Bowen [10] on the Hausdorff dimension of quasi-circles who deals with $\mathfrak{g}$ of type A_1. The first one is inspired by Bridgeman-Tayor [17] and McMullen [56], again for $\mathsf{PSL}(2,\mathbb{C})$. We note that a higher-rank version of Bowen's Theorem has been recently obtained by Farre-Pozzetti-Viaggi [23], they consider the Hausdorff dimension in the full flag variety.

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[2] due to [62] together with Bridgeman-Canary-Labourie-S. [14]

Actually we can be much more precise. Zariski closures of Hitchin representations have been classified (Guichard [33], S. [68]) so we can also look at the *intermediate strata*. It turns out that the most subtle situation is actually the Fuchsian case, which we now explain, the complete picture can be found on § 17.

Corollary (Corollary 17.2)**.** *Let* $v \in \mathsf{T}_\delta \mathcal{H}_{\mathfrak{g}}(S)$ *be tangent to a Fuchsian representation* δ*. If* $v \in \mathsf{T}^e_\delta$ *and* $\varkappa^e \subset \ker \sigma$*, then* $\operatorname{Hess}_\delta \operatorname{Hff}_\sigma(\mathsf{J}v) = 0$*. If* $\mathfrak{g}$ *has classical type then the converse is also true: if* $\operatorname{Hess}_\delta \operatorname{Hff}_\sigma(\mathsf{J}v) = 0$ *then* $v \in \bigoplus_{e:\varkappa^e \subset \ker \sigma} \mathsf{T}^e_\delta$*.*

For example, when $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{R})$ with the standard Cartan subspace $\mathfrak{a} = \{a \in \mathbb{R}^d : \sum a_i = 0\}$ and simple roots $\sigma_j(a) = a_j - a_{j+1}$; the above corollary reduces the question of understanding $\operatorname{Hess}_\delta \operatorname{Hff}_{\sigma_j} \mathsf{J}v = 0$ to describing the triples of integers (d,e,j) such that $\varkappa^e \subset \ker \sigma_j$. This condition can be rephrased in terms of an explicit Diophantine equation (see Equation (14.8)). For example, understanding degenerations on the second Grassmannian $\zeta^2(\partial\pi_1 S) \subset \operatorname{Gr}_2(\mathbb{R}^d) \subset \operatorname{Gr}_2(\mathbb{C}^d)$ reduces to the (elementary) equation

$$d - 1 = \frac{e(e+1)}{2},$$

so $\operatorname{Hess}\operatorname{Hff}_{\sigma_2}$ is only degenerate when $d = 4, 7, 11, 16...$ However, for the remaining Grassmannians the equation is more involved. For $\operatorname{Gr}_3(\mathbb{C}^d)$ the equation becomes

$$e^4 - 6de^2 + 2e^3 + 6d^2 - 6de + 11e^2 - 18d + 10e + 12 = 0,$$

which turns out to be a genus-1 complex curve, whose integer solutions can be completely described via the Elliptic logarithm method $\mathfrak{Ellog}$ (see for example Tzanakis [76]), so we obtain in § 17.2:

Corollary 1.4. *Consider* $v \in \mathsf{T}_\rho \mathcal{H}_{\mathfrak{sl}(d,\mathbb{R})}(S)$*, then one has* $\operatorname{Hess}_\rho \operatorname{Hff}_{\sigma_3}(\mathsf{J}v) = 0$ *if and only if one of the following holds :*

- $d = 6$*,* ρ *has values in* $\mathsf{PSp}(6,\mathbb{R})$ *and* $v \in \mathsf{T}^2_\rho \oplus \mathsf{T}^4_\rho$*,*
- $d = 17$*,* ρ *is Fuchsian and* $v \in \mathsf{T}^4_\rho \oplus \mathsf{T}^8_\rho$*,*
- $d = 58$*,* ρ *is Fuchsian and* $v \in \mathsf{T}^8_\rho$*.*

The Diophantine equation associated to the 4th root is

$$11d - \frac{13}{2}e - \frac{23}{3}e^2 - \frac{19}{8}e^3 - \frac{31}{24}e^4 - \frac{1}{8}e^5 - \frac{1}{24}e^6 + \frac{13}{2}de - \frac{3}{2}d^2e + 7de^2 - \frac{3}{2}d^2e^2 + de^3 + \frac{1}{2}de^4 + d^3 - 6d^2 = 6,$$

which according to Maple is a genus-4 complex curve with one singular point. To our understanding, no general method to explicitly solve this kind of equation over $\mathbb{Z}$ is known.

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2. Preliminaries

Throughout this paper we will let G be a Zariski-connected semi-simple real-algebraic group of the non-compact type with Lie algebra $\mathfrak{g}$. We will also let V be a finite-dimensional real vector space, Γ a semi-group and Γ a finitely generated word-hyperbolic group.

2.1. **Notations from Lie theory.** Let us fix $o : \mathfrak{g} \to \mathfrak{g}$ a Cartan involution with associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and let $\Phi \subset \mathfrak{a}^*$ be the set of restricted roots of $\mathfrak{a}$ in $\mathfrak{g}$. For $\alpha \in \Phi$ let us denote by

$$\mathfrak{g}_\alpha = \{u \in \mathfrak{g} : [a, u] = \alpha(a)u \,\forall a \in \mathfrak{a}\} \tag{2.1}$$

its associated root space. One has the (restricted) root space decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_0$ is the centralizer of $\mathfrak{a}$. Fix a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$ and let Φ^+ and Δ be, respectively, the associated sets of positive roots and of simple roots.

Let us denote by $(\cdot, \cdot)$ the Killing form of $\mathfrak{g}$, its restriction to $\mathfrak{a}$, and its associated dual form in the dual of $\mathfrak{a}$, $\mathfrak{a}^*$. For $\chi, \psi \in \mathfrak{a}^*$ define

$$\langle \chi, \psi \rangle = 2\frac{(\chi, \psi)}{(\psi, \psi)}.$$

The *Weyl group* of Φ, denoted by W, is the group generated by, for each $\alpha \in \Phi$, the reflection $r_\alpha : \mathfrak{a}^* \to \mathfrak{a}^*$ on the hyperplane $\alpha^\perp$,

$$r_\alpha(\chi) = \chi - \langle \chi, \alpha \rangle \alpha.$$

It is a finite group with a unique *longest* element w_0 (w.r.t. the word metric on the generating set $\{r_\alpha : \alpha \in \Delta\}$). This longest element sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$.

The *restricted weight lattice* is defined by

$$\Pi = \{\varphi \in \mathfrak{a}^* : \langle \varphi, \alpha \rangle \in \mathbb{Z} \ \forall \alpha \in \Phi\},$$

it is spanned by the *fundamental weights*: $\{\varpi_\sigma : \sigma \in \Delta\}$ where ϖ_σ is defined by

$$\langle \varpi_\sigma, \beta \rangle = d_\sigma \delta_{\sigma\beta}$$

for every $\sigma, \beta \in \Delta$, where $d_\sigma = 1$ if $2\sigma \notin \Phi^+$ and $d_\sigma = 2$ otherwise. The set Π_+ of *dominant restricted weights* is defined by $\Pi_+ = \Pi \cap (\mathfrak{a}^+)^*$.

If $\mathfrak{g}$ has *reduced* root system, then it is customary to denote by

$$\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha = \sum_{\sigma \in \Delta} \varpi_\sigma,$$

where the last equality can be found in Humphreys [38, §13.3]. For every $\sigma \in \Delta$ one has $\langle \delta, \sigma \rangle = 1$ ([38, §10.2]).

2.2. **Some $\mathfrak{sl}_2$'s of $\mathfrak{g}$.** For $\varphi \in \mathfrak{a}^*$ let $u_\varphi \in \mathfrak{a}$ be such that for all $v \in \mathfrak{a}$ one has

$$\varphi(v) = (u_\varphi, v).$$

For $\alpha \in \Phi$ let $h_\alpha \in \mathfrak{a}$ be defined such that, for all $\varphi \in \mathfrak{a}^*$ one has

$$\varphi(h_\alpha) = \langle \varphi, \alpha \rangle.$$

The vectors u_α and h_α are related by the simple formula $h_\alpha = 2u_\alpha/(u_\alpha, u_\alpha)$. Recall that for $x \in \mathfrak{g}_\alpha$ one has $[x, o(x)] = (x, o(x))u_\alpha$. Thus, for each $\alpha \in \Phi^+$ and $\mathsf{x}_\alpha \in \mathfrak{g}_\alpha$ there exists $\mathsf{y}_\alpha \in \mathfrak{g}_{-\alpha}$ such that

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto \mathsf{x}_\alpha, \quad \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto \mathsf{y}_\alpha \ \text{ and } \ \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto h_\alpha,$$

is a Lie algebra isomorphism between $\mathfrak{sl}_2(\mathbb{R})$ and the span of $\{\mathsf{x}_\alpha, \mathsf{y}_\alpha, h_\alpha\}$. Let us fix such a choice of x_α and y_α from now on.

One says that $\mathfrak{g}$ is *split* if the complexification $\mathfrak{a} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g} \otimes \mathbb{C}$. Equivalently, $\mathfrak{g}$ is split if the centralizer $\mathfrak{z}_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$ vanishes.

Assume that $\mathfrak{g}$ is split. Following Kostant [46, §5], consider the dual basis of $\{u_\sigma : \sigma \in \Delta\}$ relative to $(\cdot,\cdot)$: $(\epsilon_\alpha, u_\beta) = \delta_{\alpha\beta}$, and let $\epsilon_0 = \sum_{\sigma\in\Delta} \epsilon_\sigma \in \mathfrak{a}$. Upon writing

$$2\epsilon_0 = \sum_{\sigma\in\Delta} r_\sigma u_\sigma$$

for some $r_\sigma \in \mathbb{Z}_{>0}$, define $e^+ = \displaystyle\sum_{\sigma\in\Delta} \mathsf{x}_\sigma$ and $e^- = \displaystyle\sum_{\sigma\in\Delta} r_\sigma \mathsf{y}_\sigma$. Since $(2\epsilon_0, u_\sigma) = 2$ for every $\sigma \in \Delta$, the identification

$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \mapsto e^+, \quad \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \mapsto e^- \text{ and } \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \mapsto 2\epsilon_0,$$

is also a Lie algebra isomorphism between $\mathfrak{sl}_2(\mathbb{R})$ and the span $\mathfrak{s}$ of $\{e^+, e^-, 2\epsilon_0\}$. The Lie algebra $\mathfrak{s}$ is called *a principal* $\mathfrak{sl}_2$ and the $\operatorname{Inn}\mathfrak{g}$-conjugacy class of this representation is called *the principal* $\mathfrak{sl}_2$ of $\mathfrak{g}$.

2.3. **Cartan decomposition.** Let $\mathsf{K} \subset \mathsf{G}$ be a compact group that contains a representative for every element of the Weyl group W. This is to say, such that the normalizer $N_\mathsf{G}(\mathsf{A})$ verifies $N_\mathsf{G}(\exp\mathfrak{a}) = (N_\mathsf{G}(\exp\mathfrak{a}) \cap \mathsf{K})\exp\mathfrak{a}$. Cartan's decomposition asserts the existence of a map

$$\mu : \mathsf{G} \to \mathfrak{a}^+$$

such that for every $g \in \mathsf{G}$ there exist $k, l \in \mathsf{K}$ such that $g = k\exp(\mu(g))l$ and such that for every $g_1, g_2 \in \mathsf{G}$ one has $g_1 \in \mathsf{K}g_2\mathsf{K}$ if and only if $\mu(g_1) = \mu(g_2)$. It is called the *Cartan projection* of G.

2.4. **Jordan decomposition.** Recall that the Jordan decomposition states that every $g \in \mathsf{G}$ can be written as a commuting product $g = g_e g_h g_n$, where g_e is elliptic, g_h is diagonalizable over $\mathbb{R}$ and g_n is unipotent. The component g_h is conjugate to an element $z_g \in \exp(\mathfrak{a}^+)$ and we let

$$\lambda(g) = \mu(z_g) \in \mathfrak{a}^+.$$

The map $\lambda : \mathsf{G} \to \mathfrak{a}^+$ is called *the Jordan projection* of G.

2.5. **Flag manifolds of G.** A subset $\vartheta \subset \Delta$ determines a pair of opposite parabolic subgroups P_ϑ and $\check{\mathsf{P}}_\vartheta$ whose Lie algebras are defined by

$$\mathfrak{p}_\vartheta = \bigoplus_{\sigma\in\Phi^+\cup\{0\}} \mathfrak{g}^\sigma \oplus \bigoplus_{\sigma\in\langle\Delta-\vartheta\rangle} \mathfrak{g}^{-\sigma},$$

$$\check{\mathfrak{p}}_\vartheta = \bigoplus_{\sigma\in\Phi^+\cup\{0\}} \mathfrak{g}^{-\sigma} \oplus \bigoplus_{\sigma\in\langle\Delta-\vartheta\rangle} \mathfrak{g}^{\sigma}.$$

The group $\check{\mathsf{P}}_\vartheta$ is conjugated to the parabolic group $\mathsf{P}_{\mathrm{i}\,\vartheta}$. We denote the *flag space* associated to ϑ by $\mathcal{F}_\vartheta = \mathsf{G}/\mathsf{P}_\vartheta$. The G orbit of the pair $([\mathsf{P}_\vartheta], [\check{\mathsf{P}}_\vartheta])$ is the unique open orbit for the action of G in the product $\mathcal{F}_\vartheta \times \mathcal{F}_{\mathrm{i}\,\vartheta}$ and is denoted by $\mathcal{F}^{(2)}_\vartheta$.

Remark 2.1. If $g \in \mathsf{G}$ is such that $\sigma(\lambda(g)) > 0$ for all $\sigma \in \vartheta$ then g acts proximally on $\mathcal{F}_\vartheta$.We will denote by $(g^+, g^-) \in \mathcal{F}^{(2)}_\theta(\mathsf{G})$ the corresponding *attracting and repelling flags*, so that every flag $y \in \mathcal{F}_\theta(\mathsf{G})$ in general position with g^- verifies $g^n y \to g^+$.

2.6. **The center of the Levi group $\mathsf{P}_\vartheta \cap \check{\mathsf{P}}_\vartheta$.** We now consider the vector subspace

$$\mathfrak{a}_\vartheta = \bigcap_{\sigma \in \Delta - \vartheta} \ker \sigma.$$

Denoting by $W_\vartheta = \{w \in W : w(v) = v \quad \forall v \in \mathfrak{a}_\vartheta\}$ the subgroup of the Weyl group generated by reflections associated to roots in $\Delta - \vartheta$, there is a unique projection $\pi_\vartheta : \mathfrak{a} \to \mathfrak{a}_\vartheta$ invariant under W_ϑ.

The dual $(\mathfrak{a}_\vartheta)^*$ is canonically identified with the subspace of $\mathfrak{a}^*$ of π_ϑ-invariant linear forms. Such space is spanned by the fundamental weights of roots in ϑ,

$$(\mathfrak{a}_\vartheta)^* = \{\varphi \in \mathfrak{a}^* : \varphi \circ \pi_\vartheta = \varphi\} = \langle \varpi_\sigma | \mathfrak{a}_\vartheta : \sigma \in \vartheta \rangle. \tag{2.2}$$

We will denote, respectively, by

$$\mu_\vartheta = \pi_\vartheta \circ \mu : \mathsf{G} \to \mathfrak{a}_\vartheta$$
$$\lambda_\vartheta = \pi_\vartheta \circ \lambda : \mathsf{G} \to \mathfrak{a}_\vartheta,$$

the compositions of the Cartan and Jordan projections with π_ϑ.

2.7. **Representations of G.** The standard references for the following are Fulton-Harris [26], Humphreys [38] and Tits [74].

Let $\phi : \mathsf{G} \to \mathsf{GL}(V)$ be a finite dimensional rational[3] representation. We also denote by $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ the Lie algebra homomorphism associated to ϕ.

If $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ is irreducible then we say that $\phi : \mathsf{G} \to \mathsf{GL}(V)$ is *strongly irreducible*. This is equivalent to $\phi(\mathsf{G})$ not preserving a finite collection of subspaces of V.

The *weight space* associated to $\chi \in \mathfrak{a}^*$ is the vector space

$$V^\chi = \{v \in V : \phi(a)v = \chi(a)v \ \forall a \in \mathsf{A}\}.$$

We say that $\chi \in \mathfrak{a}^*$ is a *restricted weight* of ϕ if $V\chi \neq 0$. Tits [74, Theorem 7.2] states that the set of weights has a unique maximal element with respect to the partial order $\chi \succ \psi$ if $\chi - \psi$ is a $\mathbb{N}$-linear combination of positive roots. This is called *the highest weight* of ϕ and denoted by χ_ϕ. By definition, for every $g \in \mathsf{G}$ one has

$$\lambda_1\big(\phi(g)\big) = \chi_\phi\big(\lambda(g)\big), \tag{2.3}$$

where λ_1 is the logarithm of the spectral radius of $\phi(g)$.

We denote by $\Pi(\phi)$ the set of restricted weights of the representation ϕ

$$\Pi(\phi) = \big\{\chi \in \mathfrak{a}^* : V\chi \neq \{0\}\big\},$$

these are all bounded above by χ_ϕ (see for example Humphreys [38, §13.4 Lemma B]), namely every weight $\chi \in \Pi(\phi)$ has the form

$$\chi_\phi - \sum_{\sigma \in \Delta} n_\sigma \sigma \text{ for } n_\sigma \in \mathbb{N}.$$

Definition 2.2. Let $\phi : \mathsf{G} \to \mathsf{PGL}(V)$ be a representation. We denote by ϑ_ϕ the set of simple roots $\sigma \in \Delta$ such that $\chi_\phi - \sigma$ is still a weight of ϕ. Equivalently

$$\vartheta_\phi = \big\{\sigma \in \Delta : \langle \chi_\phi, \sigma \rangle \neq 0\big\}. \tag{2.4}$$

[3]Namely a rational map between algebraic varieties.

We denote by $\| \, \|_\phi$ an Euclidean norm on V invariant under $\phi\mathsf{K}$ and such that $\phi \exp \mathfrak{a}$ is self-adjoint, see for example Benoist-Quint's book [5, Lemma 6.33]. By definition of χ_ϕ and $\| \, \|_\phi$, and Equation (2.3) one has, for every $g \in \mathsf{G}$, that

$$\log \|\phi g\|_\phi = \chi_\phi(\mu(g)). \tag{2.5}$$

Here, with a slight abuse of notation, we denote by $\| \, \|_\phi$ also the induced operator norm, which doesn't depend on the scale of $\| \, \|_\phi$.

Denote by W_{χ_ϕ} the $\phi\mathsf{A}$-invariant complement of V_{χ_ϕ}. The stabilizer in G of W_{χ_ϕ} is $\check{\mathsf{P}}_{\vartheta_\phi}$, and thus one has a map of flag spaces

$$(\zeta_\phi, \zeta_\phi^*) : \mathcal{F}^{(2)}_{\vartheta_\phi}(\mathsf{G}) \to \mathrm{Gr}^{(2)}_{\dim V_{\chi_\phi}}(V). \tag{2.6}$$

This is a proper embedding which is an homeomorphism onto its image. Here, as above, $\mathrm{Gr}^{(2)}_{\dim V_{\chi_\phi}}(V)$ is the open $\mathsf{PGL}(V)$-orbit in the product of the Grassmannian of $(\dim V_{\chi_\phi})$-dimensional subspaces and the Grassmannian of $(\dim V - \dim V_{\chi_\phi})$-dimensional subspaces. One has the following proposition (see also Humphreys [39, Chapter XI]).

Proposition 2.3 (Tits [74])**.** *For each $\sigma \in \Delta$ there exists a finite dimensional rational irreducible representation $\phi_\sigma : \mathsf{G} \to \mathsf{PSL}(V_\sigma)$, such that χ_{ϕ_σ} is an integer multiple $l_\sigma \varpi_\sigma$ of the fundamental weight and $\dim V_{\chi_{\phi_\sigma}} = 1$.*

We will fix from now on such a set of representations and call them, for each $\sigma \in \Delta$, the *Tits representation associated to σ*.

2.8. **Gromov product and representations.** Consider $\vartheta \subset \Delta$. Recall from [66] that the *Gromov product* is the map

$$(\cdot|\cdot) : \mathcal{F}^{(2)}_\vartheta \to \mathfrak{a}_\vartheta$$

defined by the unique vector $(x|y) \in \mathfrak{a}_\vartheta$ such that

$$\varpi_\alpha\big((x|y)\big) = -\log \sin \measuredangle_{\phi_\alpha o}(\xi_{\phi_\alpha} x, \xi^*_{\phi_\alpha} y) = -\log \frac{|\varphi(v)|}{\|\varphi\|_{\phi_\alpha o} \|v\|_{\phi_\alpha o}} \geq 0$$

for all $\alpha \in \vartheta$, where $v \in \xi_{\phi_\alpha} x - \{0\}$ and $\ker \varphi = \xi^*_{\phi_\alpha} y$.

Remark 2.4. Note that

$$\max_{\alpha \in \vartheta} \varpi_\alpha((x|y)_o) = -\log \min_{\alpha \in \vartheta} \sin \measuredangle_{\phi_\alpha o}(\xi_{\phi_\alpha} x, \xi^*_{\phi_\alpha} y). \tag{2.7}$$

Note also that, since $\{\varpi_\alpha|\mathfrak{a}_\vartheta\}_{\alpha \in \vartheta}$ is a basis of $\mathfrak{a}_\vartheta$, the right hand side of equation (2.7) is comparable to the norm $\|(x|y)\|$.

The Gromov product is independent on the choice of Tits's representations of G, moreover, it keeps track of Gromov products for all irreducible representations of G:

Remark 2.5. Let $\phi : \mathsf{G} \to \mathsf{PSL}(V)$ be a proximal irreducible representation. If $(x, y) \in \mathcal{F}^{(2)}_{\vartheta_\phi}$ then[4]

$$\big(\Xi_\phi x | \Xi^*_\phi y\big)_{\phi o} = \chi_\phi\big((x|y)\big) = \sum_{\alpha \in \vartheta_\phi} \langle \chi_\phi, \sigma \rangle \, \varpi_\sigma\big((x|y)\big).$$

Note that by Definition 2.2, the coefficients in the last equation are all strictly positive.

[4]We have identified $\mathfrak{a}_{\{\alpha_1\}}$ with $\mathbb{R}$ via ϖ_{α_1}.

Proposition 2.6 (Bochi-Potrie-S. [8, Prop. 8.12])**.** *Given ϑ there exist c and c' so that for all $(x,y)\in\mathcal{F}_\vartheta^{(2)}$ one has*

$$\frac{1}{c}\|(x|y)\|\leq\inf\left\{\|\mu(g)\|:g([\mathsf{P}^\vartheta],[\check{\mathsf{P}}^\vartheta])=(x,y)\right\}\leq c\|(x|y)\|+c'.$$

2.9. **Additive cross ratio.** Let us consider the space $\mathcal{F}_\vartheta^{(4)}$ of pairs $(x,y),(z,t)\in\mathcal{F}_\vartheta^{(2)}$ with the extra transversality condition that both pairs (x,t) and (z,y) are also in general position. The (*aditive*) *cross ratio* is the map $\mathfrak{C}_\vartheta:\mathcal{F}_\vartheta^{(4)}\to\mathfrak{a}_\vartheta$ defined by

$$\mathfrak{C}_\vartheta(x,y,z,t)=(x|y)-(x|t)+(z|t)-(z|y).$$

It is G-invariant.

2.10. **Proximality.** Recall that $g\in\operatorname{End}(V)$ is *proximal* if it has a unique eigenvalue with maximal modulus and that the multiplicity if this eigenvalue in the characteristic polynomial of g is 1. The associated eigenline is denoted by $g^+\in\mathbb{P}(V)$ and g^- is its g-invariant complementary subspace.

Let $g\in\operatorname{End}(V)$ be proximal. We consider $\beta_g\in V^*$ and $v_g\in g^+$ such that $\ker\beta_g=g^-$ and $\beta_g(v_g)=1$, we also let $\pi_g(w)=\beta_g(w)v_g$ be the projection over g^+ following the decomposition $V=g^-\oplus g^+$. We finally let $V_2(g)$ be the generalized eigenspace of g associated to the second (in modulus) eigenvalue and τ_g be the only projection over $V_2(g)$ whose kernel is g-invariant.

Definition 2.7. If $g,h\in\operatorname{End}(V)$ are both proximal, we say that they are *transversally proximal* if

$$\beta_g(v_h)\beta_h(v_g)\neq0.$$

Using notation from § 2.9, one has then $\mathfrak{C}_1(g^+,g^-,h^+,h^-)=\log|\beta_g(v_h)\beta_h(v_g)|$. We further say that g and h are *strongly transversally proximal* if $\beta_h(\tau_g v_h)\neq0$.

We say then that $g\in\mathsf{G}$ is ϑ-*proximal* if for every $\sigma\in\vartheta$ one has $\phi_\sigma(g)$ is proximal. In this situation, there exists a pair $(g_\vartheta^-,g_\vartheta^+)\in\mathcal{F}_\vartheta^{(2)}$, defined by, for every $\sigma\in\vartheta$, $\Xi_{\phi_\sigma}(g_\vartheta^+)=\phi_\sigma(g)^+$, and every flag $x\in\mathcal{F}_\vartheta$ in general position with g_ϑ^- verifies $g^nx\to g_\vartheta^+$.

Let us say that two elements $g,h\in\mathsf{G}$ are *transversally ϑ-proximal* if they are ϑ-proximal and moreover $\{(g_\vartheta^+,h_\vartheta^-),(h_\vartheta^+,g_\vartheta^-)\}\subset\mathcal{F}_\vartheta^{(2)}$. One has the following result from Benoist [3].

Theorem 2.8 (Benoist [3])**.** *Let $g,h\in\mathsf{G}$ be transversally ϑ-proximal then,*

$$\lim_{n\to\infty}\lambda_\vartheta(g^nh^n)-\lambda_\vartheta(g^n)-\lambda_\vartheta(h^n)=\mathfrak{C}_\vartheta(g_\vartheta^-,g_\vartheta^+,h_\vartheta^-,h_\vartheta^+)=:\mathfrak{G}_\vartheta(g,h).$$

Lemma 2.9 (Benoist-Quint [5, Lemma 7.10])**.** *Let $g,h\in\mathsf{G}$ be loxodromic, then there exists a non-empty Zariski-open subset G_{gh} of G^2 such that whenever $(f,q)\in\mathrm{G}_{gh}$ the limit*

$$\lim_{\min\{n,m\}\to\infty}\lambda(g^mfh^nq)-m\lambda(g)-n\lambda(h)$$

exists. If g and h are transversally loxodromic, then $(\mathrm{id},\mathrm{id})\in\mathrm{G}_{gh}$.

Lemma 2.10. *Let $g\in\mathsf{G}$ be loxodromic, then there exists a non-empty open subset G_g of G^2 such that whenever $(f,q)\in\mathrm{G}_g$ one has*

$$\lim_{n\to\infty}\lambda(fg^nq)-n\lambda(g)$$

exists.

Proof. The proof follows the exact same lines as Lemma 2.9. Indeed the equation

$$\big(f(g^+), q(g^-)\big) \notin \mathcal{F}^{(2)}_{\Delta}(\mathsf{G})$$

with variables f and q, is described by polynomials, so its complement G_g is a (non-empty) Zariski-open set of G^2.

We commence by writing, for every $\sigma \in \Delta$,

$$\phi_\sigma(g^n) = \Big(\pm \exp\big(\varpi_\sigma(\lambda(g))\big)\Big)^n \pi_{g,\sigma} + \check{P}_n,$$

where $\pi_{g,\sigma}$ is the projection with image $\zeta_\sigma(g^+)$ and kernel $\zeta^*_\sigma(g^-)$ and where the spectral radius of $\check{P}_n$ is $\leq \exp(n(\varpi_\sigma - \sigma)(\lambda(g)))$. Thus, if $(f,q) \in \mathrm{G}_g$ one has

$$\lim_{n\to\infty} \frac{\phi_\sigma\big(fg^n q\big)}{(\pm)^n \exp(\varpi_\sigma(n\lambda(g)))} = f\pi_{g,\sigma}q.$$

By the condition on (f,q), one has $\mathrm{Trace}(f\pi_{g,\sigma}q) \neq 0$, thus for big enough n, $\phi_\sigma(fg^nq)$ is proximal together with the above convergence we get that $\lambda_1\phi_\sigma\big((fg^nq)\big) - n\varpi_\sigma(\lambda(g))$ converges as $n \to \infty$. Since this holds $\forall \sigma \in \Delta$ the Lemma follows. $\square$

It is also useful to consider a quantified version of proximality. Given r, ε positive we say that g is (r,ε)-*proximal* if it is proximal,

$$\|(g^-|g^+)\| \leq r$$

and for every $x \in \mathcal{F}_\vartheta$ with $\|(g^-|x)\| \leq \varepsilon^{-1}$ one has $d_{\mathcal{F}_\vartheta}(gx, g^+) \leq \varepsilon$. The following is from Benoist [2, Corollaire 6.3], a proof can also be found in S. [65, Lemma 5.6].

Theorem 2.11 (Benoist [2])**.** *For every $\delta > 0$ there exist $r, \varepsilon > 0$ such that if $g \in \mathsf{G}$ is (r,ε)-proximal then*

$$\big\|\mu_\vartheta(g) - \lambda_\vartheta(g) + (g^-_\vartheta|g^+_\vartheta)\big\| \leq \delta.$$

Lemma 2.12. *There exists C only depending on G such that for every ϑ-proximal $g \in \mathsf{G}$ and a flag $B \in \mathcal{F}_\vartheta$ transverse to g^- one has*

$$d(gB, g^+) \leq Ce^{(g^-|B)}e^{-\min\{\sigma(\lambda(g)):\sigma\in\vartheta\} + \|(g^-|g^+)\|}.$$

Proof. Follows from Theorem 2.11 together with, for example, Bochi-Potrie-S. [8, Lemma A.6]. $\square$

We record the following lemma that will be needed in § 6.

Lemma 2.13.

- *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be a strongly irreducible representation and let $\{W_k\}_{k\in F} \subset V$ be a finite collection of subspaces of V, none of which coincides with V. Consider a non-vanishing $v \in V$, then the set $\{g \in \mathsf{G} : \phi(g)v \notin \bigcup_{k\in F} W_k\}$ is Zariski-open and non-empty.*
- *Consider now a finite collection of strongly irreducible representations $\{\phi_i : \mathsf{G} \to V_i\}_{i\in I}$. For each $i \in I$ assume we have a non-vanishing $v_i \in V_i$ and a non-empty finite collection of strict subspaces $\{W^i_k : k \in F_i\}$ on V_i. Then there exists $g \in \mathsf{G}$ such that*

$$\forall i \in I \ gv_i \notin \bigcup_{k\in F_i} W_k.$$

Proof. The second item follows readily from the first: since G is connected, it is Zariski-connected and thus it is Zariski-irreducible. It follows then that a finite intersection of Zariski-open non-empty subsets is non-empty. The first item follows easily as, G being connected, irreducibility of the representations implies no finite collection of subspaces is preserved. □

2.11. **Zariski-dense sub-semigroups of G.** Let $\Lambda < \mathsf{G}$ be a semi-group. Its *limit cone* is

$$\mathscr{L}_\Lambda = \overline{\{\mathbb{R}_+\lambda(g) : g \in \Lambda\}} \subset \mathfrak{a}^+. \tag{2.8}$$

One has the following fundamental result:

Theorem 2.14 (Benoist [3, 4])**.** *Let $\Lambda < \mathsf{G}$ be a Zariski-dense sub-semi-group, then $\mathscr{L}_\Lambda$ is convex and has non-empty interior. Moreover, the group spanned by $\{\lambda(g) : g \in \Lambda\}$ is dense in $\mathfrak{a}$.*

We will moreover need the following:

Proposition 2.15 (Benoist [3, Proposition 5.1])**.** *Let $\Lambda < \mathsf{G}$ be a Zariski-dense sub-semi-group and let $\mathscr{C} \subset \mathscr{L}_\Lambda$ be a closed convex cone with non-empty interior. Then there exists a Zariski-dense sub-semi-group $\Lambda' < \Lambda$ such that $\mathscr{L}_{\Lambda'} = \mathscr{C}$. If $\mathscr{C}$ is* i*-invariant then Λ' can be furthermore chosen to be a group.*

One has moreover the following fact, that is not in the statement by Benoist but is a consequence of its proof (see the proof of Benoist [3, Lemma 4.3]):

Remark 2.16. Moreover, if $g \in \Lambda$ is loxodromic and $\mathscr{C}$ is a convex closed cone with non-empty interior and $\lambda(g) \in \mathscr{C} \subset \mathscr{L}_\Lambda$, the semi-group Λ' from the statement can be chosen to be a Schottky semi-group that contains a high enough power of g.

2.12. **Thermodynamics.** We begin by recalling some facts on Thermodynamical formalism over hyperbolic systems developed by Bowen, Ruelle, Parry, Pollicott among others, see for example [11] and Parry-Pollicott [58].

We will work on the setting of *Metric-Anosov flows.* They are a metric version of hyperbolic flows. The former are called Smale flows by Pollicott [59], who transferred to this more general setting the classical theory carried out for the latter. As we will not really make use of the definition we refer the reader to, for example, Pollicott [59], Bridgeman-Canary-Labourie-S. [14] or S. [67, Definition 2.2.1].

Let X be a compact metric space equipped with a continuous flow $\phi : X \to X$. The space of ϕ-invariant probability measures on X is denoted by $\mathcal{M}^\phi$. The *metric entropy* of $m \in \mathcal{M}^\phi$ will be denoted by $h(\phi, m)$. Via the variational principle, we will define the *pressure* of a function $f : X \to \mathbb{R}$ as

$$P(f) = \sup_{m \in \mathcal{M}^\phi} h(\phi, m) + \int_X f dm. \tag{2.9}$$

A probability measure m realizing the least upper bound is called an *equilibrium state* of f.

Two continuous maps $f, g : X \to V$ are *Livšic-cohomologous* if there exists a $U : X \to V$, of class C^1 in the direction of the flow[5], such that for all $x \in X$ one has

$$f(x) - g(x) = \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x).$$

[5]i.e. such that if for every $x \in X$, the map $t \mapsto U(\phi_t x)$ is of class C^1, and the map $x \mapsto \left.\frac{\partial}{\partial t}\right|_{t=0} U(\phi_t x)$ is continuous

The *f-period* of a periodic orbit τ with period $p_\phi(\tau)$ is

$$\ell_\tau(f) = \int_\tau f = \int_0^{p(\tau)} f(\phi_s x) ds \in V.$$

Let $f : X \to \mathbb{R}_{>0}$ be continuous. For every $x \in X$ the function $\kappa_f : X \times \mathbb{R} \to \mathbb{R}$, defined by $\kappa(x,t) = \int_0^t f(\phi_s x)ds$, is an increasing homeomorphism of $\mathbb{R}$. There is thus a continuous function $\alpha : X \times \mathbb{R} \to \mathbb{R}$ such that for all $(x,t) \in X \times \mathbb{R}$,

$$\alpha\big(x, \kappa(x,t)\big) = \kappa\big(x, \alpha(x,t)\big) = t.$$

The *reparametrization* of ϕ by $f : X \to \mathbb{R}_{>0}$ is the flow $\phi^f = (\phi^f_t : X \to X)_{t\in\mathbb{R}}$ defined, for all $(x,t) \in X \times \mathbb{R}$ by

$$\phi^f_t(x) = \phi_{\alpha(x,t)}(x).$$

The *Abramov transform* of $m \in \mathcal{M}^\phi$ is the probability measure $m^\# \in \mathcal{M}^{\phi^f}$ defined by

$$m^\# = \frac{f \cdot m}{\int f dm}. \tag{2.10}$$

We assume from now on that ϕ is a Hölder-continuous metric-Anosov. In this situation, a classical result by Livšic asserts that the Livšic-cohomology class of a Hölder-continuous function is uniquely determined by its periods. Moreover, if f is real-valued and Hölder-continuous, it has a unique equilibrium state, denoted it by ν_f. We also let $\mu_\phi = \nu_0$ be the unique probability measure maximizing entropy of ϕ.

Recall that the co-variance is defined, for Hölder-continuous $g, h : X \to \mathbb{R}$ with $0 = \int g d\nu_f = \int h d\nu_f$, by

$$\operatorname{covar}_f(g,h) := \lim_{t\to\infty} \int_X \frac{1}{t} \left(\int_0^t g(\phi_s x)ds\right)\left(\int_0^t h(\phi_s x)ds\right) d\nu_f,$$

and the variance by $\operatorname{var}_f(g) = \operatorname{covar}_f(g,g) \geq 0$.

Theorem 2.17 (Parry-Pollicott [58, Prop. 4.10,4.11])**.** *Let $f, g : X \to \mathbb{R}$ be Hölder continuous. Then one has*

$$\left.\frac{\partial P(f+tg)}{\partial t}\right|_{t=0} = \int g d\nu_f.$$

Moreover, If $\int g d\nu_f = 0$ then

$$\left.\frac{\partial^2 P(f+tg)}{\partial t^2}\right|_{t=0} = \operatorname{var}_f(g),$$

and if $\operatorname{var}_f(g) = 0$ then g is Livšic-cohomologous to zero.

The space of Hölder-continuous functions with exponent α is naturally a Banach space, and by the previously mentioned result by Livšic, the space of Livšic-cohomologically trivial functions is a closed subspace. The quotient $\text{Hölder}^\alpha(X)/\sim$ is thus a Banach space. Since P is invariant under Livšic-cohomology, Proposition 2.17 equips the space of (classes of) pressure zero functions for a fixed Hölder exponent

$$\mathcal{P}^\alpha(X) = \left\{ f \in \text{Hölder}^\alpha(X)/\sim : P(-f) = 0 \right\}$$

with a natural Riemannian metric: if $g \in \mathsf{T}_f\mathscr{P}(X) = \{g : \int g d\nu_f = 0\}$ then we let

$$\mathbf{P}_f(g) := \frac{\mathrm{var}_f(g)}{\int f d\nu_f}.$$

Since we are only considering functions up to Livšic-cohomology, Proposition 2.17 implies that $\mathbf{P}$ is non-degenerate.

We now turn to some concepts of Bridgeman-Canary-Labourie-S. [14]. Similar concepts where also earlier defined by Bonahon [9], Bridgeman [13], Burger [18], Croke-Fathi [20], Fathi-Flaminio [24], Katok-Knieper-Weiss [42], Knieper [45] and McMullen [56].

Let $f : X \to \mathbb{R}$ be Hölder-continuous and positive, consider, for $t > 0$, the finite set $\mathsf{R}_t(f) = \left\{\tau \text{ periodic} : \ell_f(\tau) \le t\right\}$ and define the *entropy of* f by

$$\hbar^f = \lim_{t\to\infty} \frac{1}{t} \log \#\mathsf{R}_t(f).$$

It is the topological entropy of ϕ^f and $\mu_{\phi^f} = \nu^{\#}_{-\hbar^f f}$ (see S. [65]).

If $g : X \to \mathbb{R}$ is also Hölder continuous then Bridgeman-Canary-Labourie-S. [14] define it's *dynamical intersection* with f by

$$\mathbf{I}_f(g) = \mathbf{I}(f,g) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}_t(f)} \sum_{\tau\in\mathsf{R}_t(f)} \frac{\ell_g(\tau)}{\ell_f(\tau)}.$$

Remark 2.18. These objects are more easily understood in terms of the reparametrization ϕ^f of ϕ by f. Then the f-period of τ is the period $p_{\phi_f}(\tau)$, $\hbar^f$ is the topological entropy of ϕ^f and, denoting by μ_{ϕ^f} its probability measure of maximal entropy then,

$$\mathbf{I}_f(g) = \int \frac{g}{f} d\mu_{\phi^f}.$$

In particular $\ker \mathbf{I}_f = \mathsf{T}_{-\hbar^f f}\mathscr{P}(X)$.

The functions $\hbar$ and $\mathbf{I}$ are well defined and vary analytically on $\text{Hölder}^\alpha(X, \mathbb{R}_+)$ and $\text{Hölder}^\alpha(X, \mathbb{R}_+) \times \text{Hölder}^\alpha(X, \mathbb{R})$ respectively. If g is also positive then we define its *normalized dynamical intersection* with f by

$$\mathbf{J}_f(g) = \mathbf{J}(f,g) = \frac{\hbar^g}{\hbar^f}\mathbf{I}_f(g).$$

We have the following two rigidity results: one global and one infinitesimal.

Theorem 2.19 (Bridgeman-Canary-Labourie-S. [14])**.** *One has* $\mathbf{J}(f,g) \ge 1$ *and equality holds only if for every periodic orbit* τ *one has*

$$\hbar^f \ell_f(\tau) = \hbar^g \ell_g(\tau).$$

Let $(f_t)_{t\in(-\varepsilon,\varepsilon)} \in \text{Hölder}^\alpha(X, \mathbb{R}_+)$ *be a* C^2 *curve with* $f_0 = f$, *then* $\mathrm{Hess}_f\, \mathbf{J}_f(\vec{f}, \vec{f}) = \mathbf{P}_f(\vec{f})$, *in particular* $\mathrm{Hess}_f\, \mathbf{J}(\vec{f}, \vec{f}) = 0$ *f and only if for every periodic orbit* τ *one has*

$$\frac{\partial}{\partial t}\Big|_{t=0} \hbar^{f_t} \ell_{f_t}(\tau) = 0, \tag{2.11}$$

or equivalently, $\partial^{\log}\hbar$ *and* $\partial^{\log} f$ *are Livšic-cohomologous w.r.t. the flow* ϕ^f.

In the above we have denoted by $\partial^{\log}$ the logarithmic derivative at 0

$$\partial^{\log} g = \frac{(\partial/\partial t)|_{t=0} g_t}{g_0}.$$

We record the following consequence of $\mathbf{J}_f(\cdot)$ being critical at f, giving the following formula for the variation of entropy, to be compared with Katok-Knieper-Weiss [42].

Corollary 2.20 ([14])**.** *Let $(f_t)_{t\in(-\varepsilon,\varepsilon)}$ be a C^1 curve of Hölder-continuous positive functions with $f_0 = f$ and denote by $\hbar_t = \hbar^{f_t}$, then*

$$\partial^{\log}\hbar = -\int \partial^{\log} f d\mu_{\phi^f} = -\frac{\int \vec{f} d\nu_{-\hbar f}}{\int f d\nu_{-\hbar f}}.$$

Assumption A. Let now $F : X \to V$ be Hölder-continuous and assume the vector space spanned by the periods of F is V. Assume moreover that F and 1 are Livšic-cohomologically independent.

The compact convex subset of V

$$\mathcal{M}^{\phi}(F) = \Big\{ \int_X F d\mu : \mu \in \mathcal{M}^{\phi} \Big\}$$

has hence non-empty interior. On the other hand, for each $\varphi \in V^*$ one can consider the pressure of the function $\varphi(F) : X \to \mathbb{R}$:

$$\mathrm{P}(\varphi) = P(-\varphi \circ F).$$

Proposition 2.17 implies that $\mathrm{P} : V^* \to \mathbb{R}$ is analytic and strictly convex. Moreover, using the natural identification $(V^*)^* = V$, one has, for $\varphi \in V^*$ that

$$d_\varphi \mathrm{P} = \int F d\nu_{-\varphi(F)}. \tag{2.12}$$

One has the following:

Proposition 2.21 (Babillot-Ledrappier [1, Prop. 1.1])**.** *Under Assumption A, the map $\wp : V^* \to V$ defined by $\varphi \mapsto d_\varphi \mathrm{P}$ is a diffeomorphism between V^* and the interior of $\mathcal{M}^{\phi}(F)$.*

Observe that our Assumption A is slightly weaker than that of Babillot-Ledrappier [1], however this does not affect the proof of [1, Prop. 1.1].

Remark 2.22. By Proposition 2.21 one has $p_F := \wp(0) = \int F d\mu_\phi \in \operatorname{int} \mathcal{M}^{\phi}(F)$.

2.13. **Anosov representations.** Anosov representations where introduced by Labourie [48] for fundamental groups of closed negatively curved manifolds and extended to arbitrary (finitely generated) word-hyperbolic groups by Guichard-Wienhard [35]. They have, since then, been object of numerous works. We will present here a very summarized situation based on [48], [35], Guéritaud-Guichard-Kassel-Wienhard [32], Kapovich-Leeb-Porti [40] and Bochi-Potrie-S. [8].

Let Γ be a finitely generated group and denote, for $\gamma \in \Gamma$, by $|\gamma|$ the word length w.r.t. a fixed finite symmetric generating set of Γ.

Definition 2.23. Let $\vartheta \subset \Delta$ be non-empty then, a representation $\rho : \Gamma \to \mathsf{G}$ is *ϑ-Anosov* if there exist c, μ positive such that for all $\gamma \in \Gamma$ and $\sigma \in \vartheta$ one has

$$\sigma\Big(\mu\big(\rho(\gamma)\big)\Big) \geq \mu|\gamma| - c. \tag{2.13}$$

We will denote by $\mathfrak{A}_\vartheta(\Gamma, \mathsf{G}) \subset \mathfrak{X}(\Gamma, \mathsf{G})$ the space of ϑ-Anosov characters. If $\mathsf{G} = \mathsf{SL}(d, \mathbb{K})$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, a $\{\sigma_1\}$-Anosov representation is commonly called *projective Anosov*.

If follows readily that for every $\sigma \in \vartheta$ the representation $\phi_\sigma \circ \rho : \Gamma \to \mathsf{GL}(V_\sigma)$ is projective-Anosov. The Theorem below can also be found in Bochi-Potrie-S. [8].

Theorem 2.24 (Kapovich-Leeb-Porti [40])**.** *If $\rho : \Gamma \to \mathsf{G}$ is ϑ-Anosov then Γ is word-hyperbolic.*

The group Γ has thus a *Gromov-boundary* $\partial\Gamma$ and a *space of geodesics*

$$\partial^2\Gamma = \{(x, y) \in (\partial\Gamma)^2 : x \neq y\}.$$

The following Proposition can be found in Bochi-Potrie-S. [8, Lemma 4.9], Kapovich-Leeb-Porti [40] and Guéritaud-Guichard-Kassel-Wienhard [32] and relates Definition 2.23 to Labourie's original definition.

Proposition 2.25. *If $\rho : \Gamma \to \mathsf{G}$ is ϑ-Anosov then there exist ρ-equivariant Hölder-continuous maps*

$$\xi^\vartheta : \partial\Gamma \to \mathcal{F}_\vartheta(\mathsf{G}) \quad \textit{and} \quad \xi^{\mathrm{i}\,\vartheta} : \partial\Gamma \to \mathcal{F}_{\mathrm{i}\,\vartheta}(\mathsf{G})$$

such that if $x, y \in \partial\Gamma$ are distinct, then $(\xi^\vartheta(x), \xi^{\mathrm{i}\,\vartheta}(y)) \in \mathcal{F}^{(2)}_\vartheta$. Moreover, if $\gamma \in \Gamma$ is hyperbolic, then $\rho(\gamma)$ is ϑ-proximal with attracting point $\xi^\vartheta(\gamma^+) = \rho(\gamma)^+_\vartheta$.

By Labourie [48] and Guichard-Wienhard [35] $\mathfrak{A}_\vartheta(\Gamma, \mathsf{G})$ is an open subset of the character variety. We also let $\mathfrak{A}^{*\mathrm{Z}}_\vartheta(\Gamma, \mathsf{G})$ be the space of regular points ρ such that $\rho(\Gamma)$ is Zariski-dense in G. Since we are assuming that G is connected, one has the following:

Proposition 2.26 (Bridgeman-Canary-Labourie-S. [14, Proposition 7.3])**.** *Assume ϑ contains at least one simple root of each factor of G. Then the space $\mathfrak{A}^{*,\mathrm{Z}}_\vartheta(\Gamma, \mathsf{G})$ is an analytic manifold.*

For surface groups one has the following description of the regular points of characters, that can be found on Labourie's book [49, § 5], from which we borrow the terminology of *very regular points*:

$$\hom^{\mathrm{vr}}(\pi_1 S, \mathsf{G}) = \big\{\rho : \mathsf{Z}_\mathsf{G}\big(\rho(\pi_1 S)\big) = \mathsf{Z}(\mathsf{G})\big\}.$$

Observe that since G is connected, morphisms with Zariski-dense image are very regular.

Theorem 2.27. *Both* $\hom^{\mathrm{vr}}(\pi_1 S, \mathsf{G})$ *and* $\hom^{\mathrm{vr}}(\pi_1 S, \mathsf{G})/\mathsf{G}$ *are analytic manifolds.*

2.13.1. *A reference flow for a group admitting an Anosov representation.* Assume that Γ admits a ϑ-Anosov representation ρ_0, onto some G and some non-empty ϑ. Fix $\sigma \in \vartheta$. Then, the composed representation $\phi_\sigma \circ \rho_0 : \Gamma \to \mathsf{GL}(V_\sigma)$ is projective Anosov and carries thus two maps

$$\xi^1 : \partial\Gamma \to \mathbb{P}(V_\sigma) \ \text{ and } \ \xi^{d-1} : \partial\Gamma \to \mathbb{P}\big(V^*_\sigma\big)$$

such that for every $(x, y) \in \partial^2\Gamma$ one has $\ker \xi^{d-1}(x) \oplus \xi^1(y) = V_\sigma$. We use the equivariant maps to construct a bundle $\mathbb{R} \to \widetilde{\mathsf{F}} \to \partial^2\Gamma$ whose fiber at $(x, y) \in \partial^2\Gamma$ is

$$\widetilde{\mathsf{F}}_{(x,y)} = \big\{(\varphi, v) \in \xi^{d-1}(x) \times \xi^1(y) : \varphi(v) = 1\big\}/\sim,$$

where $(\varphi, v) \sim (-\varphi, -v)$. This bundle is equipped with a Γ-action $\gamma(\varphi, v) = \big(\varphi \circ \rho(\gamma)^{-1}, \rho(\gamma)v\big)$ and an $\mathbb{R}$-action $\big(\widetilde{\phi}_t : \widetilde{\mathsf{F}} \to \widetilde{\mathsf{F}}\big)_{t\in\mathbb{R}}$ defined by $\widetilde{\phi}_t \cdot (\varphi, v) = (e^t\varphi, e^{-t}v)$. Let

$$\mathsf{U}\Gamma = \Gamma\backslash\widetilde{\mathsf{F}}$$

and denote by $\phi = \big(\phi_t : \mathsf{F} \to \mathsf{F}\big)_{t\in\mathbb{R}}$ the induced flow on the quotient (it is usually called *the geodesic flow of* $\Phi_\sigma \circ \rho_0$).

Theorem 2.28 (Bridgeman-Canary-Labourie-S. [14])**.** *The above* Γ*-action is properly discontinuous and co-compact. The flow* ϕ *is Hölder-continuous and metric-Anosov.*

The flow-space $\mathsf{U}\Gamma$ and flow ϕ will be fixed from now on and used as a reference flow. The set of hyperbolic elements of Γ will be denoted by Γ_{h} and for $\gamma \in \Gamma_{\mathrm{h}}$ we will denote by $\ell(\gamma)$ the period for ϕ of the periodic orbit $[\gamma]$ associated to γ.

2.13.2. *The Ledrappier potential.* We now recall a combination of facts from Bridgeman-Canary-Labourie-S. [14], Potrie-S. [60] and S. [65, 67]. Recall from the previous subsection that a base flow is fixed for Γ.

Theorem 2.29. *Let* $\rho : \Gamma \to \mathsf{G}$ *be* ϑ*-Anosov. Then there exists a Hölder-continuous map* $\mathcal{J}_\rho : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$ *such that for every* $\gamma \in \Gamma_{\mathrm{h}}$

$$\ell_{[\gamma]}(\mathcal{J}_\rho) = \lambda^\gamma_\vartheta(\rho).$$

Moreover, if $\{\rho_u : \Gamma \to \mathsf{G}\}_{u\in D}$ *is an analytic family of* ϑ*-Anosov representations, then the map* $u \mapsto \mathcal{J}_{\rho_u}$ *is analytic.*

Definition 2.30. The function $\mathcal{J}_\rho$ will be called the *Ledrappier potential of* ρ.

Let us consider the ϑ-limit cone of ρ defined by

$$\mathcal{L}_{\vartheta,\rho} = \overline{\{\mathbb{R}_+ \cdot \lambda_\vartheta(\rho(\gamma)) : \gamma \in \Gamma\}} = \mathbb{R}_+ \cdot \mathcal{M}^\phi(\mathcal{J}_\rho) \subset \mathfrak{a}_\vartheta. \tag{2.14}$$

Elements of $\operatorname{int}(\mathcal{L}_{\vartheta,\rho})^*$ will be called *length functionals*. Indeed (see S. [67, Lemma 3.4.2]), $\psi \in \operatorname{int}(\mathcal{L}_{\vartheta,\rho})^*$ if and only if $\psi(\mathcal{J}_\rho)$ is Livšic-cohomologous to a strictly positive function, or equivalently there exist μ positive such that for every $\gamma \in \Gamma$

$$\psi^\gamma(\rho) \geq \mu\ell(\gamma).$$

We can thus define the *entropy* of ψ as $\hbar^\psi_\rho = \hbar^{\psi(\mathcal{J}\rho)}$ and the *critical hypersurface*

$$\mathcal{Q}_{\vartheta,\rho} = \Big\{\psi \in \operatorname{int}(\mathcal{L}_{\vartheta,\rho})^* : \hbar^\psi_\rho = 1\Big\}. \tag{2.15}$$

It follows that $\mathcal{Q}_{\vartheta,\rho}$ is a closed co-dimension-one analytic sub-manifold that bounds a convex set, which is strictly convex if $\rho(\Gamma)$ is Zariski-dense, see S. [67, § 5.9 and 5.10] and [14], where details and more information can be found.

The Ledrappier potential embeds the space of ϑ-Anosov representations in the space of Hölder-continuous potentials Hölder$(\mathsf{U}\Gamma, \mathfrak{a}_\vartheta)$

$$\mathcal{J} : \mathfrak{A}_\vartheta(\Gamma, \mathsf{G}) \to \text{Hölder}(\mathsf{U}\Gamma, \mathfrak{a}_\vartheta)$$
$$\rho \mapsto \mathcal{J}_\rho,$$

and by Bridgeman-Canary-Labourie-S. [14] it is a real analytic map. Its differential at v is (a Livšic-cohomlogy class of) a Hölder-continuous map

$$\vec{\mathcal{J}}_v : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$$

whose periods are, by definition, $\displaystyle\int_{[\gamma]} \vec{\mathcal{J}}_v = \mathrm{d}\lambda^\gamma_\vartheta(v).$

By continuity of the ϑ-limit cone w.r.t to the representation (Theorem 2.29 and Eq. (2.14)), every $\psi \in (\mathfrak{a}_\vartheta)^*$ defines an open subset

$$\mathcal{U}_\psi = \{\eta \in \mathfrak{A}_\vartheta(\Gamma, \mathsf{G}) : \psi \in \operatorname{int}(\mathcal{L}_{\vartheta,\eta})^*\} \tag{2.16}$$

and gives in turn, with pre-composition with the Ledrappier potential, a map $\mathcal{U}_\psi \to$ Hölder$(\mathsf{U}\Gamma, \mathbb{R})$, $\rho \mapsto \psi(\mathcal{J}_\rho)$.

Definition 2.31. The *ψ-pressure form* $\mathbf{P}^\psi$ on $\mathcal{U}_\psi$ is the pullback of the pressure metric on $\mathscr{P}(\mathsf{U}\Gamma)$ by the map $\rho \mapsto \hbar^\psi_\rho \cdot \psi(\mathcal{J}_\rho)$.

For $\psi \in \mathcal{Q}_{\vartheta,\rho}$ and $\eta \in \mathcal{U}_\psi$, we can define the ψ-dynamical intersection of the pair (ρ, η) as

$$\mathbf{I}^\psi(\rho,\eta) = \mathbf{I}\big(\psi(\mathcal{J}_\rho), \psi(\mathcal{J}_\eta)\big) = \lim_{t\to\infty} \frac{1}{\#\mathsf{R}^\psi_t(\rho)} \sum_{\gamma \in \mathsf{R}^\psi_t(\rho)} \frac{\psi\big(\lambda(\eta(\gamma))\big)}{\psi\big(\lambda(\rho(\gamma))\big)}, \tag{2.17}$$

where $\mathsf{R}^\psi_t(\rho) = \big\{\gamma \in \Gamma_{\mathrm{h}} : \psi^\gamma(\rho)\big) \leq t\}$. By Theorem 2.19, upon denoting by $\mathbf{J}^\psi$ the associated normalized intersection, one has

$$\mathrm{Hess}_\rho\, \mathbf{J}^\psi_\rho = \mathbf{P}^\psi_\rho.$$

Recall that the opposition involution i of $\mathfrak{a}$ is induced by an external automorphism $\underline{i} : \mathsf{G} \to \mathsf{G}$, which acts whence on characters $\underline{i} : \mathfrak{X}(\Gamma, \mathsf{G}) \to \mathfrak{X}(\Gamma, \mathsf{G})$. From Equation (2.17) one concludes:

Corollary 2.32. *For every $\psi \in \mathcal{Q}_{\vartheta,\rho}$ and $\eta \in \mathcal{U}_\psi$ one has $\mathbf{I}^\psi(\rho,\eta) = \mathbf{I}^\psi(\underline{i}\rho, \underline{i}\eta)$. In particular, the involution $\underline{i}$ on $\mathfrak{X}(\Gamma,\mathsf{G})$ is an isometry of any pressure form $\mathbf{P}^\psi$.*

2.13.3. *Entropy regulating form.* We introduce the set of *ψ-normalized variations* of v by

$$\mathbb{V}^\psi_{\vartheta,v} = \overline{\left\{\frac{\mathrm{d}\lambda^\gamma_\vartheta(v)}{\psi\big(\lambda^\gamma(\rho)\big)} : \gamma \in \Gamma_{\mathrm{h}}\right\}}.$$

Remark 2.33. By Eq. (2.11), the Pressure form $\mathbf{P}^\psi_\rho$ degenerates at v if and only if $\psi(\mathbb{V}^\psi_{\vartheta,v}) = \partial^{\log} \hbar^\psi$, in particular $\mathbb{V}^\psi_{\vartheta,v}$ is contained on a level set of ψ.

Lemma 2.34. *If ρ is ϑ-Anosov then $\mathbb{V}^\psi_{\vartheta,v}$ is compact and convex. Moreover the map $v \mapsto \mathbb{V}^\psi_{\vartheta,v}$ is continuous on the open set $\mathsf{T}\mathcal{U}_\psi$.*

Proof. Recall we have a base-flow $\phi = (\phi_t : \mathsf{U}\Gamma \to \mathsf{U}\Gamma)_{t\in\mathbb{R}}$. Consider the Ledrappier potential $\mathcal{J} : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$ together with its variation $\vec{\mathcal{J}} : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$ associated to v. By definition, for every $\gamma \in \Gamma$

$$\int_\gamma \mathcal{J} = \lambda^\gamma(\rho) \text{ and } \int_\gamma \vec{\mathcal{J}} = \mathrm{d}\lambda^\gamma_\vartheta(v).$$

If we let $\phi^{\psi(\mathcal{J})}$ be the reparametrization of ϕ by $\psi(\mathcal{J})$ (recall § 2.12) and we let $\mu \mapsto \mu^\#$ be the Abramov transform on invariant measures, then

$$\frac{\mathrm{d}\lambda^\gamma_\vartheta(v)}{\psi^\gamma(\rho)} = \frac{1}{\ell_{\phi^{\psi(\mathcal{J})}}(\gamma)} \int_{\gamma^\#} \frac{\vec{\mathcal{J}}}{\psi(\mathcal{J})}$$

so

$$\mathbb{V}^\psi_{\vartheta,v} = \mathcal{M}^{\phi^{\psi(\mathcal{J})}}\big(\vec{\mathcal{J}}/\psi(\mathcal{J})\big), \tag{2.18}$$

which yields the desired conclusion. ∎

It is convenient to name a particular point of $\mathbb{V}^\psi_{\vartheta,v}$, to do so we introduce the *length cone-bundle*:

$$\mathfrak{L}_\vartheta(\Gamma,\mathsf{G}) = \Big\{(\psi,\rho) \in \mathfrak{a}_\vartheta \times \mathfrak{A}_\vartheta(\Gamma,\mathsf{G}) : \psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*\Big\}.$$

Corollary 2.35 (Entropy regulating form)**.** *There exists an analytic fibered map*

$$p : \mathfrak{L}_\vartheta(\Gamma, \mathsf{G}) \to \mathsf{T}^*\mathfrak{A}_\vartheta(\Gamma, \mathsf{G}) \otimes \mathfrak{a}_\vartheta$$

such that if $\psi \in \mathfrak{L}_\vartheta(\Gamma, \mathsf{G})$ *and* $v \in \mathsf{T}_\rho\mathfrak{A}(\Gamma, \mathsf{G})$ *then* $p_\psi(v) \in \text{rel-int}\, \mathbb{V}^{\psi}_{\vartheta,v}$ *and*

$$\psi\big(p_\psi(v)\big) = -\mathrm{d}\log \hbar^{\psi}(v). \tag{2.19}$$

Moreover, if $\vec{\mathcal{J}}$ *and* $\psi(\mathcal{J})$ *are Livšic-cohomologically independent, then* $\mathbb{V}^{\psi}_{\vartheta,v}$ *has non-empty interior and thus* $p_\psi(v) \in \text{int}\, \mathbb{V}^{\psi}_{\vartheta,v}$.

Proof. For $(\psi, \rho) \in \mathfrak{L}_\vartheta(\Gamma, \mathsf{G})$ and $v \in \mathsf{T}_\rho\mathfrak{A}_\vartheta(\Gamma, \mathsf{G})$ we let $p_\psi v = \int \frac{\vec{\mathcal{J}}}{\psi(\mathcal{J})} d\mu_{\phi^\psi} \in \mathbb{V}^{\psi}_{\vartheta,v} \subset \mathfrak{a}_\vartheta$. Then the statements of the Corollary follow readily from the combination of Corollary 2.20, Proposition 2.21 and Remark 2.22. ☐

2.13.4. *A needed Lemma.* The following is only needed in Corollary 11.5.

Lemma 2.36. *Let* G *be semi-simple of non-compact type. Fix a simple root* $\alpha \in \Delta$*, let* $\rho \in \mathfrak{A}_{\{\alpha\}}(\Gamma, \mathsf{G})$ *have Zariski-dense image and consider a non-zero* $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$*. Then, for any* $\psi \in \text{int}\,(\mathcal{L}_{\{\alpha\},\rho})^*$ *one has* $\mathbb{V}^{\psi}_{\{\alpha\},v}$ *has non-empty interior and* $p_\psi v \in \text{int}\, \mathbb{V}^{\psi}_{\{\alpha\},v}$.

Proof. All elements in $(\mathfrak{a}_{\{\alpha\}})^*$ are scalar multiples of the fundamental weight ϖ_α so it suffices to prove the result for $\psi = \varpi_\alpha$. In order to apply Corollary 2.35 we need to show that the hypothesis of Proposition 2.21 by Babillot-Ledrappier are verified, this is to say, we need to show then that $\vec{\mathcal{J}}/\varpi_\alpha(\mathcal{J})$ and 1 are Livšic-cohomologicaly independent. Now $\mathfrak{a}_{\{\alpha\}}$ is also 1-dimensional so the last statement is equivalent to prove that $\varpi_\alpha(\vec{\mathcal{J}})$ and $\varpi_\alpha(\mathcal{J})$ are independent, which is the exact content of Theorem 2.37 below. ☐

We conclude the section by stating the following result from Bridgeman-Canary-Labourie-S. [14]. If $\mathsf{G} \subset \mathsf{SL}(d, \mathbb{R})$ is semi-simple and $\rho : \Gamma \to \mathsf{G} \subset \mathsf{SL}(d, \mathbb{R})$ then, we say that ρ is G-*generic* if $\rho(\Gamma)$ contains an element that is loxodromic in G.

Theorem 2.37 ([14, Lemma 9.8+Prop. 10.1])**.** *Let* $\rho : \Gamma \to \mathsf{SL}(d, \mathbb{R})$ *be projective Anosov and such that* $\rho(\Gamma)$ *acts irreducibly on* $\mathbb{R}^d$*. Assume that* $\rho(\Gamma) \subset \mathsf{G}$ *for some semi-simple Lie group* G *and let* $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}$ *be a differentiable curve through* ρ *of* G*-generic representations. If there exists* $K \in \mathbb{R}$ *such that for all* $\gamma \in \Gamma$ *one has*

$$\frac{\partial}{\partial t}\Big|_{t=0} \lambda_1(\rho_t\gamma) = K\lambda_1(\rho\gamma)$$

then $K = 0$ *and the cocycle* $\mathsf{u} \in H^1_{\mathrm{Ad}_{\mathsf{G}}\,\rho}(\Gamma, \mathsf{G})$ *associated to* $\vec{\rho}$ *is cohomologically trivial.*

2.13.5. *Cross ratios.* A decomposition $\mathbb{R}^d = \ell \oplus V$ into a line and a hyperplane defines a rank-1 projection denoted by $\pi_{(\ell,V)}$. If $\mathbb{R}^d = r \oplus W$ is such that $\dim r = 1$ and moreover $r \cap V = \{0\} = \ell \cap W$ (we say in this case that the decompositions are *transverse*) then we define the (multiplicative) *cross ratio* by

$$\mathsf{B}_1(\ell, V, r, W) = \mathrm{Trace}(\pi_{\ell,V}\pi_{r,W}). \tag{2.20}$$

Let now $\eta : \Gamma \to \mathsf{PGL}(d, \mathbb{R})$ be a projective Anosov representation with equivariant maps $\xi^1 : \partial\Gamma \to \mathbb{P}(\mathbb{R}^d)$ and $\xi^{d-1} : \partial\Gamma \to \mathbb{P}((\mathbb{R}^d)^*)$.

A pair $(x,y)\in\partial^2\Gamma$ defines a decomposition $\mathbb{R}^d=\xi^1(y)\oplus\xi^{d-1}(x)$. If we let $\partial^{(4)}\Gamma$ the space of pairwise distinct four-tuples then an element $(x,y,z,t)\in\partial^{(4)}\Gamma$ defines two transverse decompositions of $\mathbb{R}^d$ so we define the *cross ratio map* $\mathrm{b}_\eta:\partial^{(4)}\Gamma\to\mathbb{R}$ by

$$\mathrm{b}_\eta(x,y,z,t)=\mathsf{B}_1\Big(\xi^1(y),\xi^{d-1}(x),\xi^1(t),\xi^{d-1}(z)\Big).$$

If we now let $\rho:\Gamma\to\mathsf{G}$ be ϑ-Anosov then we define for later use the $\mathfrak{a}_\vartheta$-valued cross ratio map as $\mathfrak{b}_\rho:\partial^{(4)}\Gamma\to\mathfrak{a}_\vartheta$ by, for each (x,y,z,t), as the unique vector such that for every $\sigma\in\vartheta$

$$\varpi_\sigma(\mathfrak{b}_\rho(x,y,z,t))=\log|\mathrm{b}_{\Phi_\sigma}(x,y,z,t)|.$$

Theorem 2.38 ([14])**.** *Let $\{\rho_u:\Gamma\to\mathsf{PGL}(d,\mathbb{R})\}_{u\in D}$ be an analytic family of projective-Anosov representations, then for every four-tuple $(x,y,z,t)\in\partial^{(4)}\Gamma$, the map $u\mapsto\mathrm{b}_{\rho_u}(x,y,z,t)$ is real-analytic. In particular, if $\{\rho_u:\Gamma\to\mathsf{G}\}_{u\in D}$ is an analytic family of ϑ-Anosov representations then for every four-tuple $(x,y,z,t)\in\partial^{(4)}\Gamma$ the map $u\mapsto\mathfrak{b}_{\rho_u}(x,y,,z,t)$ is real-analytic.*

Part 1. Affine actions

3. Margulis invariant: basics

3.1. An elementary lemma. Let V be a finite dimensional real vector space and consider the *affine group* $\mathsf{Aff}(V)=\mathsf{GL}(V)\ltimes V$. An element $f\in\mathsf{Aff}(V)$ has a *linear part* $\dot{f}\in\mathsf{GL}(V)$ and a *translation part* v_f so that $\forall u\in V$ one has

$$f(u)=\dot{f}(u)+v_f.$$

Let us consider the (possibly trivial) generalized eigenspace of $\dot{f}$ associated to the eigenvalue 1

$$\mathcal{O}=\mathcal{O}(\dot{f})=\big\{w\in V:\exists n\geq 0\text{ with }(\dot{f}-\mathrm{id})^n w=0\big\}$$

and define the *un-normalized Margulis invariant of f*, $\breve{\mathsf{m}}(f)\in\mathcal{O}$, as follows. Jordan's decomposition of $\dot{f}$ guarantees the existence of a $\dot{f}$-invariant decomposition $V=W\oplus\mathcal{O}$, let $\pi^1:V\to\mathcal{O}$ be the associated projection and define

$$\breve{\mathsf{m}}(f)=\pi^1(v_f). \tag{3.1}$$

Remark 3.1. It follows at once that $\breve{\mathsf{m}}(f^{-1})=-\breve{\mathsf{m}}(f)$.

Lemma 3.2. *Consider $f\in\mathsf{Aff}(V)$, then there exists $o\in V$ such that the translate $\mathcal{V}^0(f)=\mathcal{O}(\dot{f})+o$ is invariant by f. Moreover, the transformation $\mathcal{O}\to\mathcal{O}$ defined by $v\mapsto f(v+o)-o$ has linear part $\dot{f}|\mathcal{O}$ and translation part $\breve{\mathsf{m}}(f)$.*

Proof. Indeed, 1 is not an eigenvalue of $\dot{f}|W$ and thus $\dot{f}|W-\mathrm{id}_W$ is invertible. Let $\pi:V\to W$ the projection following the decomposition $V=W\oplus\mathcal{O}$. The transformation $\pi f=\pi\circ(f|W):W\to W$

$$\pi f(w)=\dot{f}w+\pi(v_f)$$

has a unique fixed point $o = o_f$, defined by $o = -(\dot f - \mathrm{id})^{-1}\pi(v_f)$. For $v \in \mathcal{O}$ one has

$$\begin{aligned} f(o+v) &= \dot f(o+v) + v_f \\ &= \dot f(o) + \dot f(v) + \pi v_f + \check{\mathsf{m}}(f) \\ &= \dot f(o) + \pi v_f + \dot f(v) + \check{\mathsf{m}}(f) \\ &= o + \dot f(v) + \check{\mathsf{m}}(f) \in o + \mathcal{O}, \end{aligned}$$

concluding the proof. □

We end this subsection with the following remarks:

Remark 3.3.
- For every $h \in \mathsf{GL}(V)$ one has $h\big(\check{\mathsf{m}}(f)\big) = \check{\mathsf{m}}(hfh^{-1})$.
- Assume that $\dot f|\mathcal{O} = \mathrm{id}$, then for all $u \in V$ the maps f and $f + u - \dot f u$ have the same un-normalized Margulis invariant, that is

$$\check{\mathsf{m}}(f) = \check{\mathsf{m}}(f + u - \dot f u).$$

  Indeed $\pi^0\big(u - \dot f(u)\big) = 0$.

3.2. **Around the 0-restricted-weight of an irreducible representation.** We consider now a reductive real-algebraic group G and an irreducible representation $\phi : \mathsf{G} \to \mathsf{SL}(V)$ with $0 \in \Pi_\phi$ and define

$$\begin{aligned} \Pi_\phi^{w_0} &= \big\{\chi \in \Pi_\phi : \chi \circ w_0 = \chi\big\}, \\ \mathcal{N} &= \bigoplus_{\chi \in \Pi_\phi^{w_0}} V^\chi. \end{aligned}$$

The vector space $\mathcal{N}$ will be called the *ideally neutral space* of ϕ. For $v \in \mathfrak{a}$ we let

$$\begin{aligned} \Pi_\phi^{v,+} &= \{\chi \in \Pi_\phi : \chi(v) > 0\}, \\ \Pi_\phi^{v,-} &= \{\chi \in \Pi_\phi : \chi(v) < 0\}. \end{aligned}$$

An element of $\mathrm{Fix}(\mathrm{i})$ necessarily annihilates all $\chi \in \Pi_\phi^{w_0}$. We fix an $X_0 \in \mathfrak{a}^+ \cap \mathrm{Fix}(\mathrm{i})$ which does not belong to the finite union of kernels $\ker \chi$, $\chi \in \Pi_\phi \setminus \Pi_\phi^{w_0}$. This choice provides a partition $\Pi_\phi = \Pi_\phi^{w_0} \cup \Pi_\phi^+ \cup \Pi_\phi^-$, where

$$\begin{aligned} \Pi_\phi^+ &:= \Pi_\phi^{X_0,+}, \\ \Pi_\phi^- &:= \Pi_\phi^{X_0,-}, \end{aligned}$$

with the advantage that $w_0(\Pi_\phi^+) = \Pi_\phi^-$ and $\{\chi \in \Pi_\phi : \chi(X_0) = 0\} = \Pi_\phi^{w_0}$.

We now consider the set of simple roots θ such

$$\Delta - \theta = \{\sigma \in \Delta : r_\sigma(\Pi_\phi^+) = \Pi_\phi^+\}.$$

Then one has the following (observe we are using the opposite convention for parabolic groups than Smilga [72]).

Proposition 3.4 (Smilga [72, Prop. 6.4, Lemma 6.5])**.** *If $\sigma \in \Delta - \theta$ then r_σ necessarily fixes each $\chi \in \Pi_\phi^{w_0}$. The set θ is* i*-invariant and the parabolic group P^θ is the stabilizer in G of each of*

$$a^+ := \bigoplus_{\chi \in \Pi_\phi^+} V^\chi \text{ and } A^+ := a^+ \oplus \mathcal{N}.$$

The group $\check{\mathsf{P}}^{\theta}$ *is the stabilizer of* $a^- := \bigoplus_{\chi\in\Pi_\phi^-} V^\chi$ *and* $A^- := a^- \oplus \mathcal{N}$.

Lemma 3.5 (Smilga [72, Lemme 7.12])**.** *Consequently, for each* $\chi \in \Pi_\phi^{w_0}$ *the algebra* $\mathfrak{g}^0 \bigoplus_{\sigma\in\langle\Delta-\theta\rangle} \mathfrak{g}^\sigma \oplus \mathfrak{g}^{-\sigma}$ *acts trivially on* V^χ *and thus the Levi group* L_θ *acts as* $\phi(\mathsf{MA})$ *on* $\mathcal{N}$.

Proof. Fix $\sigma \in \langle\Delta - \theta\rangle$, the reflection r_σ also stabilizes Π_ϕ^+. Since χ is w_0-invariant one readily sees that $\chi + \sigma$ is not. Now, $r_\sigma(\chi) \in \Pi_\phi^{w_0} \cup \Pi_\phi^-$ (or else $\chi = r_\sigma(r_\sigma\chi) \in \Pi_\phi^+$). So if $\chi + \sigma \in \Pi_\phi$ then necessarily $\chi + \sigma(X_0) > 0$, i.e. $\chi + \sigma \in \Pi_\phi^+$. However,

$$r_\sigma(\chi + \sigma)(X_0) = (r_\sigma\chi - \sigma)(X_0) < 0,$$

i.e. $r_\sigma(\chi + \sigma) \in \Pi_\phi^-$ achieving a contradiction. Whence $\chi + \sigma \notin \Pi_\phi$ and thus $\phi(\mathfrak{g}_\sigma)V^\chi \subset V^{\chi-\sigma} = \{0\}$ concluding the proof. □

We now consider the action of $\phi(\mathsf{MA})$ on $\mathcal{N}$. The first remarkable subspace, the *neutralizing space* of ϕ, is the fixed point subspace of $\phi(\mathsf{MA})$ on $\mathcal{N}$, and coincides with the fixed point set of $\phi(\mathsf{M})$ on V^0. It is denoted by

$$\mathcal{T} := V^{\phi(\mathsf{MA})} = V^{0,\phi(\mathsf{M})} = \{v \in V^0 : \phi(\mathsf{M})v = v\}.$$

Definition 3.6. The *neutralizing dimension* of ϕ is $\operatorname{neudim}\phi := \dim\mathcal{T}$.

Let $\mathcal{T}^{\perp_\mathsf{M}} \subset \mathcal{N}$ be the $\phi(\mathsf{M})$-invariant complement, the decomposition

$$\mathcal{N} = \mathcal{T}^{\perp_\mathsf{M}} \oplus \mathcal{T} \tag{3.2}$$

is canonical given ϕ. Let $\pi^{\mathcal{T}} : \mathcal{N} \to \mathcal{T}$ the associated projection.

Example 3.7.

- If G is split then $\mathcal{T} = V^0$, see for example Ghosh [28, Lemma 2.5].
- For the defining representation of $\mathsf{SO}_{1,n}$ the neutralizing space is trivial.
- If we let $\phi = \operatorname{Ad} : \mathsf{G} \to \mathsf{SL}(\mathfrak{g})$ be the adjoint representation then $\mathcal{N} = \mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}$, and writing $\mathfrak{m} = [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m})$ one has that M preserves each factor $\mathfrak{g}^0 = [\mathfrak{m},\mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ and acts trivially on $\mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ so

$$\mathcal{T} = \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}.$$

  The typical situation when $\mathfrak{Z}(\mathfrak{m}) \neq \{0\}$ is when G is a complex group considered as a real group, in which case $\mathfrak{Z}(\mathfrak{m}) = \mathfrak{m}$ and $\mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$ (as a complex Lie algebra).

If we let $k = \dim a^+$ then, as $\phi(w_0)a^+ = a^-$, one has $k = \dim a^-$, let also n be the dimension of $\mathcal{N}$, and denote by $\mathrm{F}_{k,n}$ the space of partial flags (u_k, u_n) consisting on a k-dimensional subspace u_k of V contained on an n-dimensional one u_n. Proposition 3.4 and Equation (2.6) provide an algebraic ϕ-equivariant map

$$\begin{aligned} : \mathcal{F}_\theta(\mathsf{G}) &\to \mathrm{F}_{k,n}(V) \\ x &\mapsto (x_\phi, X_\phi) \end{aligned} \tag{3.3}$$

such that if $(x,y) \in \mathcal{F}_\theta^{(2)}(\mathsf{G})$ then (x_ϕ, X_ϕ) and (y_ϕ, Y_ϕ) are in general position.

Corollary 3.8. *There exists a constant* C *only depending on* ϕ *such that if* $(x,y) \in \mathcal{F}_\theta^{(2)}$ *then*

$$\|(x|y)\|\frac{1}{C} \le \big\|\big((x,X)|(y,Y)\big)\big\|_{\phi o} \le C\|(x|y)\|.$$

Proof. Follows from Proposition 3.4 and Remark 2.5. □

A pair of flags in general position $(x,y)\in\mathcal{F}^{(2)}_\theta(\mathsf{G})$ determines then an n-dimensional space, namely

$$X_\phi\cap Y_\phi.$$

By Lemma 3.5, such a space carries a natural decomposition into *its neutralizing space* and the "M-invariant complement". Indeed, if $g_0,g_1\in\mathsf{G}$ are such that $g_0(x,y)=\big([\mathsf{P}^\theta],[\check{\mathsf{P}}^\theta]\big)=g_1(x,y)$ then $g_0g_1^{-1}$ belongs to L_θ, which preserves the decomposition (3.2). Consequently, both subspaces of $X_\phi\cap Y_\phi$

$$\mathcal{T}^{(x,y)}=\phi(g_0)^{-1}\mathcal{T}$$
$$\mathcal{T}^{(x,y),\perp_{\mathsf{M}}}=\phi(g_0)^{-1}(\mathcal{T}^{\perp_{\mathsf{M}}})$$

are independent of g_0. We fix from now on, for each $(x,y)\in\mathcal{F}^{(2)}_\theta$, an element $\psi_{(x,y)}\in\mathsf{G}$ such that

$$\psi_{(x,y)}\big([\mathsf{P}^\theta],[\check{\mathsf{P}}^\theta]\big)=(x,y). \tag{3.4}$$

By Proposition 2.6 such an element can be chosen so that for $\|\mu(\psi_{(x,y)})\|$ is coarsely comparable to the norm of the Gromov product $(x|y)^\theta_o$.

3.3. **(ϕ,X_0)-compatible elements.** If $\dot g\in\mathsf{G}$ is θ-proximal we have, using Equation (3.3), a decomposition

$$V=\left(\dot g^+_\phi\oplus\dot g^-_\phi\right)\oplus\dot G^0_\phi \tag{3.5}$$

where $\dot G^0_\phi:=\dot G^+_\phi\cap\dot G^-_\phi$. Observe that

(i) $\phi(\dot g)|\dot G^0_\phi$ decomposes as sum of roto-hometheties,

(ii) the subspace $\dot g^+_\phi$ is not necessarily attracting for $\phi(\dot g)$ on the Grassmannian $\mathrm{Gr}_k(V)$. This only happens if $\lambda(\dot g)$ lies in a specific neighborhood of X_0: the open cone

$$\{v\in\mathfrak{a}^+:\chi^+(v)>\chi^0(v),\ \forall\chi^+\in\Pi^+_\phi,\chi^0\in\Pi^{w_0}_\phi\}.$$

Symmetrizing in order to deal with inverses we consider the sub-cone of $\mathfrak{a}^+$:

$$\mathcal{X}_\phi=\left\{v\in\mathfrak{a}^+:\Pi^+_\phi(v)>\Pi^{w_0}_\phi(v)>\Pi^-_\phi(v)\right\}. \tag{3.6}$$

Because of item (ii) one introduces the following definition.

Definition 3.9. An element $g\in\mathsf{G}\ltimes V$ is (ϕ,X_0)-*compatible* if $\lambda(\dot g)\in\mathcal{X}_\phi$.

Lemma 3.10. *A (ϕ,X_0)-compatible element g has θ-proximal linear part. Moreover the flag $(\dot g^+_\phi,\dot G^+_\phi)\in\mathrm{F}_{k,n}(V)$ is attracting for $\phi(\dot g)$ with repelling flag $(\dot g^-_\phi,\dot G^-_\phi)$.*

Proof. By definition, for every $\sigma\in\theta$ there exists $\chi\in\Pi^+_\phi$ such that $r_\sigma\chi\notin\Pi^+_\phi$. Thus, if g is (ϕ,X_0)-compatible one has, for such χ,

$$\begin{aligned}\chi(\lambda(\dot g))&>(r_\sigma\chi)(\lambda(\dot g))\\&=(\chi-\langle\chi,\sigma\rangle\sigma)(\lambda(\dot g))\\&=\chi(\lambda(\dot g))-\langle\chi,\sigma\rangle\sigma(\lambda(\dot g)),\end{aligned}$$

giving $\sigma(\lambda(\dot g))>0$. The second statement follows by definition. ∎

Observe that although the action $\phi(\dot g)$ on $\dot G^0_\phi = \dot G^+_\phi \cap \dot G^-_\phi$ might have some expanding/contracting spaces, this expansion is dominated by the one on $\dot g^+_\phi$ and the same holds for the contraction and $\dot g^-_\phi$. Moreover, recall notation from § 3.1,

$$\mathcal{T}^{(\dot g^+_\phi, \dot g^-_\phi)} \subset \mathcal{O}(\dot g) \subset \dot G^0_\phi.$$

Let now $g \in \mathsf{G} \ltimes V$ be (ϕ, X_0)-compatible. Equation (3.1) provides an unnormalized Margulis invariant

$$\check{\mathsf{m}}(g) \in \mathcal{O}(\dot g) \subset \dot G^0_\phi = \mathcal{T}^{(\dot g^+_\phi, \dot g^-_\phi)} \oplus \mathcal{T}^{(\dot g^+_\phi, \dot g^-_\phi), \perp_{\mathsf{M}}}$$

that we further project onto $\mathcal{T}^{(\dot g^+_\phi, \dot g^-_\phi)}$ and push to $\mathcal{T}$ via $\psi_{(\dot g^+, \dot g^-)}$ to obtain the Margulis invariant of g:

Definition 3.11. The *Margulis invariant* of a (ϕ, X_0)-compatible $g \in \mathsf{G} \ltimes V$ is

$$\mathsf{m}(g) := \pi^{\mathcal{T}}\big(\psi_{(\dot g^+, \dot g^-)}\big(\check{\mathsf{m}}(g)\big)\big) \in \mathcal{T}.$$

Remark 3.12.

- By combining both items on Remark 3.3, it is invariant under conjugation by an element of $\mathsf{G} \ltimes V$.
- Observe that the Weyl group acts on $\mathcal{T}$ and it follows from Remark 3.1 that $\mathsf{m}(g^{-1}) = w_0 \cdot \big(-\mathsf{m}(g)\big)$.

3.4. **Invariant flags.** Assume $g \in \mathsf{G} \ltimes V$ is (ϕ, X_0)-compatible and consider the projections parallel to the decomposition in Equation (3.5)

$$\pi^{\pm}_g : V \to \dot g^+_\phi \oplus \dot g^-_\phi,$$
$$\pi^0_g : V \to \dot G^0_\phi.$$

Definition 3.13. The unique fixed point of the affine map $\pi^{\pm}_g \circ g : \dot g^+_\phi \oplus \dot g^-_\phi \to \dot g^+_\phi \oplus \dot g^-_\phi$ will be denoted by o_g.

Whence, each of the following affine flags is g-invariant:

$$G^+ = \dot G^+_\phi + o_g,$$
$$G^- = \dot G^-_\phi + o_g,$$
$$G^0 = \dot G^0_\phi + o_g.$$

Moreover, inside G^+ the notion of *parallel to* $\dot g^+_\phi$ is preserved by g. Indeed

$$g(\dot g^+_\phi + o_g) = \dot g^+_\phi + o_g + \pi^0(v_g).$$

The same thing happens for G^- and $\dot g^-_\phi$.

For an affine p-dimensional subspace $U \subset V$ we consider $\dot U = \{u - v : u, v \in U\}$, it is a vector subspace of V. We also consider the distance on the space of affine p-dimensional spaces defined by

$$d_{\mathsf{Aff}}(U, W) := d_{\mathrm{Gr}_p(V)}(\dot U, \dot W) + \inf\big\{\|u - w\|_\phi : u \in U \text{ and } w \in W\big\}.$$

Lemma 3.14. *There exists a constant c only depending on ϕ so that if g is (ϕ, X_0)-compatible and B is a co-dimension k affine subspace of V transverse to $\dot g^-_\phi$ and we let $\{b\} = B \cap g^-_\phi$ then,*

$$d_{\mathsf{Aff}}(gB, G^+) \leq \frac{c}{e^{-(\dot g^-_\phi | \dot B)_{\phi o}}} e^{-\min\{\sigma(\lambda(\dot g)) : \sigma \in \theta\} + \|(\dot g^+ | \dot g^-)\|} + \|\dot g | \dot g^-_\phi\| \|b - o_g\|.$$

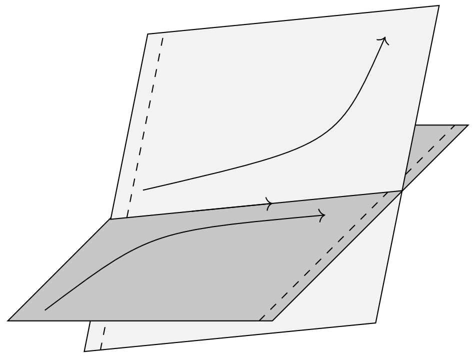


FIGURE 1. Dynamics of a (ϕ, X_0)-compatible element of $\mathsf{G} \ltimes V$.

Proof. One has $gB = \dot{g}\dot{B} + g(b) = \dot{g}\dot{B}$. Since by Lemma 3.10 $\dot{G}^+_{\phi}$ is attracting for $\dot{g}$ with repelling flag $\dot{g}^-_{\phi}$, the distance between the vector subspaces $\dot{g}\dot{B}$ and $\dot{G}^+_{\phi}$ is controlled by Corollary 3.8 together with Lemma 2.12 giving the first term in the inequality stated in the Lemma.

To control the second term, recall we have defined $\check{\mathsf{m}}(g) = \pi^0_g(v_g)$, so $o_g + \check{\mathsf{m}}(g) \in G^+$. By definition $b = v + o_g$ for some $v \in \dot{g}^-_{\phi}$, so $gb = \dot{g}v + o_g + \check{\mathsf{m}}(g)$. Whence

$$\inf \left\{ \|gv - u\| : v \in B, u \in G^+ \right\} \leq \left\| gb - \left(o_g + \check{\mathsf{m}}(g)\right) \right\| \leq \|\dot{g}v\|,$$

completing the proof. □

Remark 3.15. We remark that, as g is (ϕ, X_0)-compatible, the spectral radius of $\dot{g}|\dot{g}^-_{\phi}$ is strictly smaller than 1, so $\|\dot{g}^n|\dot{g}^-_{\phi}\| \to 0$ as $n \to \infty$. Moreover, Smilga [72, Proposition 7.26] states that given $C \geq 1$ there exists c such that every (ϕ, X_0)-compatible g with $\|(\dot{g}^+|\dot{g}^-)\| \leq C$ it holds

$$e^{-\min\{\sigma(\mu(\dot{g})):\sigma\in\theta\}} \leq c\|\dot{g}|\dot{g}^-_{\phi}\|.$$

Definition 3.16. The *affine-contraction* of a (ϕ, X_0)-compatible $g \in \mathsf{G} \ltimes V$ is

$$\varsigma(g) := \|\dot{g}|\dot{g}^-_{\phi}\| \cdot \|\dot{g}^{-1}|\dot{G}^+_{\phi}\| \cdot e^{\|o_g\|}.$$

We conclude the section with the following perturbation Lemma.

Lemma 3.17. *Let $h \in \mathsf{G} \ltimes V$ be (ϕ, X_0)-compatible and loxodromic. Then, there exists a non-empty Zariski-open subset $\mathcal{G}_h$ of G^2 such that for every pair $f, q \in \mathsf{G} \ltimes V$ with $(\dot{f}, \dot{q}) \in \mathcal{G}_h$ the following holds.*

(i) *The sequence $\mathbb{R}_+ \cdot \lambda(\dot{f}\dot{h}^n\dot{q}) \to \mathbb{R}_+ \cdot \lambda(\dot{h})$ as $n \to \infty$. In particular fh^nq is (ϕ, X_0)-compatible for all large enough n.*

(ii) *The attracting affine flag of fh^nq converges, as $n \to \infty$, to $f(H^+)$ and the repelling affine flag converges to $q^{-1}(H^-)$.*

Proof. We commence by considering the Zariski-open subset $\mathrm{G}_h \subset \mathsf{G}^2$ of Lemma 2.10. For every $(\dot{f}, \dot{q}) \in \mathrm{G}_h$ the Lemma implies

$$\frac{\lambda(\dot{f}\dot{h}^n\dot{q}) - n\lambda(\dot{h})}{n} \xrightarrow[n\to\infty]{} 0,$$

giving the first statement in item (i). Since $\dot{h}$ is loxodromic, $\lambda(h)$ lies in the interior of $\mathcal{X}_{\phi}$ and thus item (i) follows readily.

Now, if $\dot{f}, \dot{q}$ moreover verify

$$\dot{f}(\dot{H}_{\Phi}^{+}) \cap \dot{q}^{-1}(\dot{h}_{\Phi}^{-}) = \{0\} \tag{3.7}$$

then, there exists a neighbourhood $\mathcal{U}$ of $f(H^+)$ such that for all $B \in \mathcal{U}$ one has $q(B) \pitchfork \dot{h}_{\Phi}^{-}$. Lemma 3.14 implies that $h^n q(B) \to H^+$, so $fh^n q(B) \to f(H^+)$ as $n \to \infty$; i.e. if n is large enough, there is a small neighbourhood of $f(H^+)$ that is sent to itself by $fh^n q$, entailing that the attracting affine flag of $fh^n q$ is arbitrarily close to $f(H^+)$ as $n \to \infty$.

To deal with the repelling flag, the argument follows similarly by considering the equation

$$\dot{q}^{-1}(\dot{H}_{\Phi}^{-}) \cap \dot{f}(\dot{h}_{\Phi}^{+}) = \{0\}. \tag{3.8}$$

The lemma is settled by observing that, since all the flags $\dot{h}_{\Phi}^{-}, \dot{h}_{\Phi}^{+}, \dot{H}_{\Phi}^{-}$ and $\dot{H}_{\Phi}^{+}$ are fixed, Equations (3.7) and (3.8) determine Zariski-open non-empty subsets of G^2, which we further intersect with G_{hh} to conclude the proof. □

4. The Affine Ratio and Affine Limit Cone

4.1. **The Affine Ratio.**

Let us consider a real vector space V of dimension $2k+l$ and the incomplete flag space

$$\mathrm{F}_{k,l}(V) = \big\{(\ell, W) : \ell \in \mathrm{Gr}_k(V), W \in \mathrm{Gr}_{k+l}(V),\, \ell \subset W\big\}.$$

It is a self-dual flag space of V and two such flags, $(a, A), (b, B)$, are in general position if $a \cap B = b \cap A = \{0\}$, in this case $A \cap B$ has dimension l. For $v \in V$, let us denote by $(a, A) + v$ the affine flag $(a + v, A + v)$.

Consider four flags $(a^+, A^+), (a^-, A^-), (b^+, B^+), (b^-, B^-)$ in $\mathrm{F}_{k,l}(V)$, pairwise in general position, and consider also two arbitrary vectors $v, w \in V$. We will define an invariant of the four-tuple of affine flags

$$\begin{aligned}
\mathsf{a}^+ &= (a^+, A^+) + w,\\
\mathsf{a}^- &= (a^-, A^-) + w,\\
\mathsf{b}^+ &= (b^+, B^+) + v,\\
\mathsf{b}^- &= (b^-, B^-) + v,
\end{aligned}$$

which we call, by analogy with the cross ratio of four lines in the plane, the *affine ratio*. Let us denote by $A^0 = A^+ \cap A^-$ (and $B^0 = B^+ \cap B^-$). This invariant will be an affine map from $A^0 + w$ to itself,

$$\breve{z}(\mathsf{a}^-, \mathsf{b}^+, \mathsf{b}^-, \mathsf{a}^+) : A^0 \to A^0,$$

defined by following procedure.

Consider $u \in A^0$ so that $u + w \in A^0 + w \subset A^- + w$, translate $u + w$ parallel to $a^- + w$ until one reaches $A^- + w \cap (B^+ + v)$, call this vector u_1. Translate u_1 parallel to $b^+ + v$ until reaching $u_2 \in (B^+ + v) \cap (B^- + v)$, translate u_2 parallel to $b^- + v$ until reaching $u_3 \in (B^- + v) \cap A^+$, translate u_3 parallel to $a^+ + w$ until reaching again $u_4 \in (A^+ + w) \cap (A^- + w) = A^0 + w$. Then we let (see Figure 2)

$$\breve{z}(\mathsf{a}^-, \mathsf{b}^+, \mathsf{b}^-, \mathsf{a}^+)(u) := u_4 - w \in A^0.$$

Remark 4.1. The affine ratio is invariant under the action of $\mathsf{GL}(V) \ltimes V$ in the following sense: if $f \in \mathsf{GL}(V) \ltimes V$ then

$$f \breve{z}(\mathsf{a}^-, \mathsf{b}^+, \mathsf{b}^-, \mathsf{a}^+) f^{-1} = \breve{z}(f\mathsf{a}^+, f\mathsf{a}^-, f\mathsf{b}^+, f\mathsf{b}^-).$$

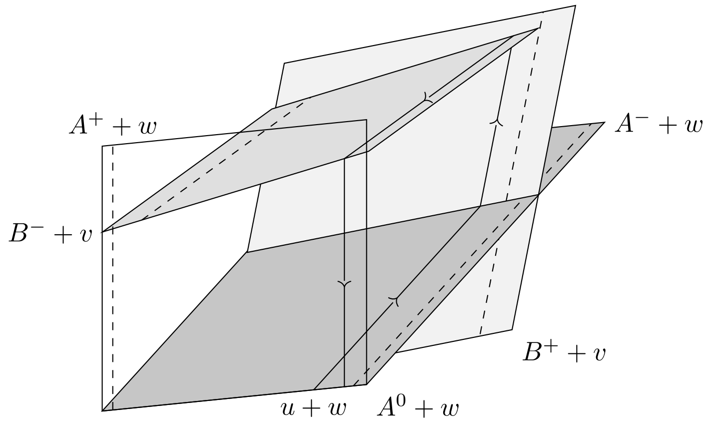


FIGURE 2. Definition of the Affine Ratio

Given a decomposition $V = U \oplus W$ let us denote by $\pi^{U,W} : V \to U$ the projection along it, so $\mathrm{id} = \pi^{U,W} \oplus \pi^{W,U}$. The next lemma computes the translation part of $\check{\mathfrak{r}}$.

Lemma 4.2. *One has* $\check{\mathfrak{r}}(\mathsf{a}^-, \mathsf{b}^+, \mathsf{b}^-, \mathsf{a}^+)(0) = \pi^{A^0, a^+ \oplus b^-} \pi^{b^+ \oplus a^-, B^0}(v - w)$.

Proof. Without loss of generality we may assume that $w = 0$. Using the definition of $\check{\mathfrak{r}}$ and the fact that $u = 0$ one directly observes

$$u_1 = \pi^{a^- B^+} v.$$

From the decomposition $u_1 - v = \pi^{b^+ B^-}(u_1 - v) + \pi^{B^- b^+}(u_1 - v)$, one obtains that

$$u_2 = u_1 - \pi^{b^+ B^-}(u_1 - v) = \pi^{b^+ B^-} v + \pi^{B^- b^+} \pi^{a^- B^+} v.$$

Now one has $u_3 = \pi^{A^+ b^-} u_2$; so finally we get

$$\begin{aligned}
\check{\mathfrak{r}}(x^+, x^-, y^+, y^-)(0) = u_4 &= \pi^{A^- a^+} u_3 = \pi^{A^- a^+} \pi^{A^+ b^-} \left(\pi^{b^+ B^-} v + \pi^{B^- b^+} \pi^{a^- B^+} v \right) \\
&= \pi^{A^0, a^+ \oplus b^-} \left(v - \pi^{B^- b^+} v + \pi^{B^- b^+} \pi^{a^- B^+} v \right) \\
&= \pi^{A^0, a^+ \oplus b^-} \left(v + \pi^{B^- b^+} (\pi^{a^- B^+} v - v) \right) \\
&= \pi^{A^0, a^+ \oplus b^-} \left(v - \pi^{B^- b^+} (\pi^{B^+ a^-} v) \right) \\
&= \pi^{A^0, a^+ \oplus b^-} \left(v - \pi^{B^- \cap B^+, b^+ \oplus a^-} v \right) \\
&= \pi^{A^0, a^+ \oplus b^-} \pi^{b^+ \oplus a^-, B^0}(v),
\end{aligned}$$

as desired. □

In particular, if all affine flags have one point in common $p = a^+ \cap a^- \cap b^+ \cap b^-$, then the translation part of $\check{\mathfrak{r}}$ is 0 since we can then then take $v = w = p$.

An *affine* θ*-flag* of $\mathsf{G} \ltimes V$ consists on the affine partial flag $(x_\Phi, X_\Phi) + v$, for some $x \in \mathcal{F}_\theta(\mathsf{G})$ and $v \in V$. Let us now consider the affine ratio as an invariant of four affine flags of the group $\mathsf{G} \ltimes V$, we will also normalize it so as to consider it as a map of $\mathcal{N}$ to itself.

Definition 4.3. Let X^+, X^-, Y^+, Y^- be a four-tuple of pairwise transverse affine θ-flags of $\mathsf{G} \ltimes V$, and define

$$\mathfrak{r}_{\mathcal{N}}(X^-, Y^+, Y^-, X^+) : \mathcal{N} \to \mathcal{N}$$

by conjugating $\breve{\mathfrak{r}}\big(X^-, Y^+, Y^-, X^+\big)$ with the map

$$\phi\big(\psi_{(x^+,x^-)}\big) \circ \mathrm{T}_{-w}$$

that sends the pairs of affine flags $X^+ = (x^+_\phi, X^+_\phi) + w$ and $X^- = (x^-_\phi, X^-_\phi) + w$ to (a^+, A^+) and (a^-, A^-). Finally, let us define the *Translation Affine Ratio*

$$\mathfrak{r}_{\mathcal{T}}(X^-, Y^+, Y^-, X^+) \in \mathcal{T}$$

as the translation part of $\mathfrak{r}_{\mathcal{N}}(X^-, Y^+, Y^-, X^+) : \mathcal{N} \to \mathcal{N}$ along the neutralizing space $\mathcal{T}$, this is to say

$$\mathfrak{r}_{\mathcal{T}}(X^-, Y^+, Y^-, X^+) := \pi^{\mathcal{T}}\big(\mathfrak{r}_{\mathcal{N}}(X^-, Y^+, Y^-, X^+)(0)\big).$$

For a subset $\Lambda < \mathsf{G} \ltimes V$ we let $\dot{\Lambda} < \mathsf{G}$ be the subset consisting on its linear parts. The purpose of this section is the following result.

Proposition 4.4. *Assume G is Zariski-connected. Let $\Lambda < \mathsf{G} \ltimes V$ be a Zariski-dense sub-semi-group such that $\mathcal{L}_{\dot{\Lambda}} \cap \mathcal{X}_\phi$ has non-empty interior. Then the set*

$$\big\{\mathfrak{r}_{\mathcal{N}}(G^-, F^+, F^-, G^+)(0) : f, g \in \Lambda \textit{ are } (\phi, X_0)\textit{-compatible and transverse}\big\}$$

is not contained in a hyperplane of $\mathcal{N}$. Consequently, the set

$$\big\{\mathfrak{r}_{\mathcal{T}}(G^-, F^+, F^-, G^+) : f, g \in \Lambda \textit{ are } (\phi, X_0)\textit{-compatible and transverse}\big\}$$

is not contained in a hyperplane of $\mathcal{T}$.

Proof. By Zariski-density, and since $\mathcal{L}_{\dot{\Lambda}} \cap \mathcal{X}_\phi$ has non-empty interior, there exists a pair $g, h \in \Lambda$ such that the corresponding linear parts $\dot{g}$ and $\dot{h}$ are (ϕ, X_0)-compatible, loxodromic and transverse.

By conjugating Λ we may, and will, assume that (recall Definition 3.13)

$$o_g = 0, \ \dot{g}^+ = [\mathsf{P}^\theta], \text{ and } \dot{g}^- = [\check{\mathsf{P}}^\theta],$$

so that $(\dot{g}^+_\phi, \dot{G}^+_\phi) = (a^+, A^+)$, $(\dot{g}^-_\phi, \dot{G}^-_\phi) = (a^-, A^-)$, $A^0 = \mathcal{N}$, $G^+ = A^+$ and $G^- = A^-$. Applying Lemma 4.2 we see that we have to understand the vector

$$\begin{aligned} \nu(g,h) &:= \pi^{\mathcal{N}, a^+ \oplus \dot{h}^-_\phi} \pi^{\dot{h}^+_\phi \oplus a^-, \dot{H}^0}(o_h) \\ &= \pi^{\mathcal{N}, a^+ \oplus \dot{h}^-_\phi}\big((a^- \oplus \dot{h}^+_\phi) \cap H^0\big), \end{aligned} \tag{4.1}$$

for different choices of h. More precisely, we fix g and h as above and a hyperplane $U \subset \mathcal{N}$, we will find $f, q \in \Lambda$ and $n \in \mathbb{N}$ such that $\nu(g, fh^n q) \notin U$.

We apply Lemma 3.17 to find a non-empty Zariski-open subset $\mathcal{G}_h \subset \mathsf{G}^2$ such that if $f, q \in \mathsf{G} \ltimes V$ verify $(\dot{f}, \dot{q}) \subset \mathcal{G}_h$ then, for all large enough n one has $fh^n q$ is (ϕ, X_0)-compatible with attracting flag arbitrary close to $f(H^+)$ and repelling flag arbitrary close to $q^{-1}(H^-)$ (as $n \to \infty$).

To find elements $f, q \in \Lambda$ such that $\nu(g, fh^n q) \notin U$ for big enough n, we use Equation (4.1) and the above descriptions of the invariant affine flags of $fh^n q$ to see that we must find $f, q, \in \Lambda$ such that $(\dot{f}, \dot{q}) \in \mathcal{G}_h$ and

$$\big(U \oplus a^+ \oplus \dot{q}^{-1}(\dot{h}^-_\phi)\big) \cap \Big(\big(a^- \oplus \dot{f}(\dot{h}^+_\phi)\big) \cap \big(f(H^+) \cap q^{-1}(H^-)\big)\Big) = \emptyset. \tag{4.2}$$

This last equation defines a Zariski-open subset of $(\mathsf{G} \ltimes V)^2$, the variables being (f, q). Since $(\mathsf{G} \ltimes V)^2$ is Zariski-connected, it is Zariski-irreducible and thus a finite collection of non-empty Zariski-open subsets has non-trivial intersection. We already now that $\mathcal{G}_h$ is non-empty, so provided (4.2) defines a non-empty set we have: since Λ is Zariski-dense in $\mathsf{G} \ltimes V$, the product semi-group $\Lambda \times \Lambda$ is Zariski-dense in $(\mathsf{G} \ltimes V)^2$ and thus a pair $(f, q) \in \Lambda^2$ with $(\dot{f}, \dot{q}) \in \mathcal{G}_h$ and satisfying Equation (4.2) will exist.

To complete the proof it remains thus to show that there exist $f, q \in \mathsf{G} \ltimes V$ that verify (4.2). Consider then $\dot{f} = \dot{q} = \mathrm{id}$. It suffices to find $v \in V$ such that

$$\left(U \oplus a^+ \oplus \dot{h}_{\phi}^-\right) \cap \left(\dot{h}_{\phi}^+ \oplus a^-\right) \cap \Big(\mathrm{T}_v\left(H^+ \cap H^-\right)\Big) = \emptyset. \tag{4.3}$$

In order to do so, we observe that

$$\dim \Big(\left(U \oplus a^+ \oplus \dot{h}_{\phi}^-\right) \cap \left(\dot{h}_{\phi}^+ \oplus a^-\right)\Big) = 2k - 1,$$
$$\dim H^+ \cap H^- = l.$$

The second equality is obvious, the first one follows since $U \oplus a^+ \oplus \dot{h}_{\phi}^-$ has co-dimension 1 and $\dot{h}_{\phi}^+ \oplus a^-$ is $2k$-dimensional (by transversality of g and h) and not contained in the former.

Thus, as the dimensions do not add up to $\dim V = 2k + l$, we can translate the latter as to not intersect the former. i.e. there exists $v \in V$ such that Equation (4.3) holds, as desired. □

4.2. **The affine ratio and additivity default of Marguils' invariants.** Given $C > 0$, we say that two (ϕ, X_0)-compatible elements $g_0, g_1 \in \mathsf{G} \ltimes V$ are a *C-transverse pair* if for all $i, j \in \{0, 1\}$ one has $\|(\dot{g}_i^+ | \dot{g}_j^-)\| \le C$.

Theorem 4.5 (Smilga)**.** *Given $\varepsilon > 0$ there exists $C > 0$ such that if $f, q \in \mathsf{G} \ltimes V$ are a (ϕ, X_0)-compatible C-transverse pair with affine contraction $\le 1/C$, then*

$$\left\|\mathsf{m}(fq) - \big(\mathsf{m}(f) + \mathsf{m}(q)\big) - \mathfrak{r}_{\mathcal{T}}(Q^-, Q^+, F^-, F^+)\right\| < \varepsilon.$$

Quick sketch of proof. The proof is essentially contained in Smilga [72, Proposition 9.3], however the statement is not exactly the same us ours so we just explain the main ideas.

We want to understand the Margulis invariant of fq by decomposing it into its factors f and q. The former is, up to normalizing its invariant flags, the translation part of fq restricted to the neutralizing space

$$\mathcal{T}^{\left((\dot{f}\dot{q})^+, (\dot{f}\dot{q})^-\right)} \subset (FQ)^0.$$

If the angles between the invariant flags of $\dot{f}$ and $\dot{q}$ are controlled, and moreover the affine contraction $\varsigma(f)$ is small enough (recall Definition 3.16), Lemma 3.14 implies that, the attracting flag of fq is close to F^+. An analogous control for quantities related to q^{-1} implies that the repelling flag of fq is arbitrarily close to Q^-, so the ideally neutral space $(FQ)^0$ of fq is close to $F^+ \cap Q^-$. Similarly qf has attracting flag close to Q^+, repelling flag close to F^-, and ideally neutral space close to $Q^+ \cap F^-$. See Figure 3.

Now, the map q conjugates fq and qf, so q sends $(FQ)^0$ so $(QF)^0$, and analogously, f sends $(QF)^0$ to $(FQ)^0$.

The idea is then to decompose the map $q : (FQ)^0 \to (QF)^0$ as the projection from $(FQ)^0$ to Q^0 parallel to q^-, followed by the projection Q^0 to $(QF)^0$ parallel to q^+ and some quasi-translation of $(FQ)^0$. Some errors have to be taken into account here as $(FQ)^0$ is not exactly $F^+ \cap Q^-$ but only close to it, these errors tend to be negligible. However, for the approximation of $q : (FQ)^0 \to (QF)^0$ with this composition of projections and a quasi-translation to be good, a control on the total separation of the spaces, namely, for example a control on $\|o_q\|$ and $\|o_f\|$ is sufficient (together with the previous control on angles/size of q and f). This is resolved in Lemmas 9.8 and 9.10 and Corollary 9.9 of Smilga [72].

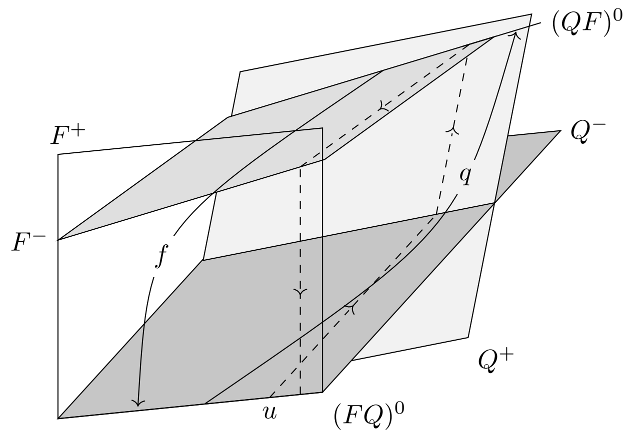


FIGURE 3. Schematic situation in Theorem 4.5, the ideally neutral spaces $(FQ)^0$ and $(QF)^0$ are very close (but do not coincide with) $F^+ \cap Q^-$ and $Q^+ \cap F^-$ respectively.

On readily sees then the Affine Ratio of Figure 2 appearing as the corresponding default, giving the result. □

4.3. **The affine limit cone is convex and has non-empty interior.** In analogy with Benoist's limit cone [3] we define the *affine limit cone* $\mathscr{A}_\Lambda$ of a Zariski dense sub-semigroup $\Lambda < \mathsf{G} \ltimes V$ as the smallest closed cone of $\mathcal{T}$ that contains the set of Margulis invariants

$$\mathscr{A}_\Lambda = \overline{\left\{\mathbb{R}_+ \cdot \mathsf{m}(f) : f \in \Lambda \text{ is } (\phi, X_0)\text{-compatible}\right\}}.$$

In light of Proposition 4.4 and Theorem 4.5 one has the following result. Similar versions will also appear in Kassel-Smilga [41] and in Ghosh [29].

Corollary 4.6. *Let $\phi : \mathsf{G} \to \mathsf{GL}(V)$ be an irreducible representation with non-trivial neutralizing space and let $\Lambda < \mathsf{G} \ltimes V$ be a Zariski-dense sub-semi-group such that* $\operatorname{int} \mathcal{L}_{\dot\Lambda} \cap \mathcal{X}_\phi \neq \emptyset$. *Then the affine limit cone $\mathscr{A}_\Lambda$ is convex and has non-empty interior.*

In contrast with Benoist's limit cone, it is fairly common for $\mathscr{A}_\Lambda$ to be the whole space $\mathcal{T}$. Indeed, this is the case when w_0 acts trivially on $\mathcal{T}$.

Proof. We first establish convexity. To that end, we fix a pair $g, h \in \Lambda$ of (ϕ, X_0)-compatible, loxodromic elements, we have to find an element in Λ whose Margulis invariant lies about $\mathbb{R}_+(\mathsf{m}(g) + \mathsf{m}(h))$.

We consider the Zariski open sets $\mathcal{G}_g$ and $\mathcal{G}_h$ given by Lemma 3.17 applied to g and h respectively. Since g and h are fixed, each of the equations

$$\begin{aligned}\dot{f}(\dot{H}^+_\phi) \cap \dot{h}^-_\phi &= \{0\},\\ \dot{H}^+_\phi \cap q^{-1}(\dot{h}^-_\phi) &= \{0\},\\ \dot{G}^-_\phi \cap \dot{q}_0^{-1}(\dot{g}^-_\phi) &= \{0\},\\ \dot{f}_0(\dot{G}^+_\phi) \cap \dot{g}^-_\phi &= \{0\},\\ \dot{H}^+_\phi \cap \dot{q}_0^{-1}(\dot{g}^-_\phi) &= \{0\},\\ \dot{G}^+_\phi \cap \dot{q}^{-1}(\dot{h}^-) &= \{0\},\end{aligned}$$

defines a Zariski-open non-empty subset of G. Since G is Zariski-irreducible finite collections of non-empty open sets intersect and thus there exist $f, q, f_0, q_0 \in \Lambda$ such that $(\dot{f}, \dot{q}) \in \mathcal{G}_h$, $(\dot{f}_0, \dot{q}_0) \in \mathcal{G}_g$, that moreover verify the corresponding equations above. For large enough n we have thus, by means of Lemma 3.17, that

- fh^nq and $f_0g^nq_0$ are (ϕ, X_0)-compatible,
- fh^nq and h are transverse,
- $f_0g^nq_0$ and g are transverse,
- for all $m \geq 1$ the elements h^mfh^nq and $g^mf_0g^nq_0$ are also transverse (and (ϕ, X_0)-compatible by Lemma 2.9).

Applying now Theorem 4.5 we see that, for big enough n,

$$\begin{aligned}\lim_{k\to\infty} \frac{\mathsf{m}(g^kf_0g^nq_0h^kfh^nq)}{k} &= \lim_{k\to\infty} \frac{\mathsf{m}(g^kf_0g^nq_0) + \mathsf{m}(h^kfh^nq)}{k}\\ &= \lim_{k\to\infty} \frac{\mathsf{m}(g^k) + \mathsf{m}(f_0g^nq_0) + \mathsf{m}(h^k) + \mathsf{m}(fh^nq)}{k}\\ &= \mathsf{m}(g) + \mathsf{m}(h),\end{aligned}$$

proving convexity of $\mathscr{A}_\Lambda$.

To prove that $\mathscr{A}_\Lambda$ has non-empty interior we use Theorem 4.5 together with Proposition 4.4. Indeed, if there exists a hyperplane $U \subset \mathcal{T}$ such that $\mathsf{m}(g) \in U$ for every (ϕ, X_0)-compatible $g \in \Lambda$, then for any transverse pair of (ϕ, X_0)-compatible elements $f, q \in \Lambda$ we have

$$\mathfrak{r}_{\mathcal{T}}(Q^-, F^+, F^-, Q^+) = \lim_{n\to\infty} \mathsf{m}(f^nq^n) - n(\mathsf{m}(f) + \mathsf{m}(q)) \in U,$$

contradicting Proposition 4.4. Finally, a convex cone that is not contained in a hyperplane has non-empty interior, completing the proof. $\square$

5. The case of reducible representations

We now consider a finite collection of non-trivial irreducible representations $\{\phi_i : \mathsf{G} \to \mathsf{SL}(V_i)\}_{i\in I}$ with $0 \in \Pi_{\phi_i}$ and study the representation

$$\phi := \bigoplus_i \phi_i : \mathsf{G} \to V := \bigoplus_i V_i.$$

Fix $X_0 \in \mathfrak{a}^+ \cap \mathrm{Fix}(\mathrm{i})$ which does not belong to the finite union of kernels $\ker \chi$, $\chi \in \Pi_{\phi_i} \setminus \Pi^{w_0}_{\phi_i}$. Let $\theta_i \subset \Delta$ be the set of simple roots stabilizing $\Pi^+_{\phi_i}$ and

$$\theta := \bigcup_{i\in I} \theta_i.$$

It we let

$$a_i^+ = \bigcup_{\chi\in\Pi^+_{\phi_i}} V^\chi \text{ and } A_i^+ = \bigcup_{\chi\in\Pi^+_{\phi_i}\cup\Pi^{w_0}_{\phi_i}} V^\chi$$

then the parabolic group P_θ is the stabilizer in G of $a^+ := \bigoplus_i a_i^+$ and of $A^+ := \bigoplus_i A_i^+$ and, if we denote by $p_i : \mathcal{F}_\theta \to \mathcal{F}_{\theta_i}$ the natural projection, then we have a map $\mathcal{F}_\theta \to \mathrm{Gr}_{\sum_i k_i, \sum_i n_i}(V)$ defined by

$$x \mapsto (x_\phi, X_\phi) := \Big(\bigoplus_i (p_i(x))_{\phi_i}, \bigoplus_i (P_i(X))_{\phi_i}\Big),$$

(recall Equation (3.3)). We let $\mathcal{X}_\phi := \bigcap_i \mathcal{X}_{\phi_i}$, where $\mathcal{X}_{\phi_i}$ is defined as in Equation (3.6), and we say that $g \in \mathsf{G} \ltimes_\phi V$ is (ϕ, X_0)-*compatible* if $\lambda(\dot g) \in \mathcal{X}_\phi$.

For each i let us denote by $\mathcal{T}_i$ the associated neutralizing space and define the *trivializing space* of ϕ by

$$\mathcal{T} = \bigoplus_i \mathcal{T}_i.$$

Define also, for a (ϕ, X_0)-compatible $g = (\dot g, v) \in \mathsf{G} \ltimes_\phi V$ its *Margulis invariant* by

$$\mathsf{m}(g) := \sum_i \mathsf{m}(\dot g, v_i),$$

where $v = \sum_i v_i$ in the decomposition $V = \bigoplus_i V_i$.

We extend the definitions of affine ratio and affine limit cone and the exact same proof of Corollary 4.6 gives the following:

Corollary 5.1. *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be as above and let $\Lambda < \mathsf{G} \ltimes V$ be a Zariski-dense sub-semi-group such that* $\operatorname{int} \mathcal{L}_{\dot\Lambda} \cap \mathcal{X}_\phi \neq \emptyset$. *Then the affine limit cone $\mathscr{A}_\Lambda \subset \mathcal{T}$ is convex and has non-empty interior.*

6. The cocycle viewpoint: Zariski density

Let us fix a (possibly reducible) representation $\phi : \mathsf{G} \to \mathsf{SL}(V)$. If $\Gamma < \mathsf{G}$ is a semi-group, a *cocycle* over ϕ is a map $\mathsf{u} : \Gamma \to V$ such that for every $\dot g, \dot h \in \Gamma$ one has

$$\mathsf{u}(\dot g \dot h) = \mathsf{u}(\dot g) + \phi(\dot g)\mathsf{u}(\dot h);$$

it is a *co-boundary* if there exists $v \in V$ with $\mathsf{u}(\dot g) = v - \phi(\dot g)v$. The vector space

$$H^1_\phi(\Gamma, V) = \frac{\{\text{cocycles over } \phi\}}{\{\text{co-boundaries}\}}$$

is called *the first twisted cohomology group.*

Semi-groups of $\mathsf{G} \ltimes_\phi V$ whose linear part is fixed and equal to Γ, are in bijective correspondence with cocycles over ϕ via

$$\mathsf{u} \mapsto \Gamma_\mathsf{u} := \{(\dot g, \mathsf{u}(\dot g)) \in \mathsf{G} \ltimes_\phi V : \dot g \in \Gamma\}.$$

Two such groups are conjugated by a pure translation, if and only if the associated cocycles differ by a co-boundary, i.e. are cohomologous.

One has the following result of Ghosh [28] that crucially uses Whitehead's Lemma on the vanishing of the first cohomology of semi-simple Lie algebra representations.

Proposition 6.1 (Ghosh [28, Proposition 6.2])**.** *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be strongly irreducible and non-trivial. Let $\Gamma < \mathsf{G}$ be Zariski-dense and $\mathsf{u} \in H^1_\phi(\Gamma, V)$ be non-trivial. Then the group Γ_u is Zariski-dense in $\mathsf{G} \ltimes_\phi V$.*

Proof. We add some details for later use. Let X be the Zariski closure of Γ_{u} and consider 'the linear part' morphism $\mathrm{L} : \mathsf{X} \to \mathsf{G}$. Since Γ is Zariski-dense in G we have that $\mathrm{L}(\mathsf{X})$ is surjective. Thus for every $g \in \mathsf{G}$ there exists $u_g \in V$ such that $(g, u_g) \in \mathsf{X}$.

If L were injective then such u_g would be unique giving a well defined cocycle, $g \mapsto u_g \in V$ of G over ϕ, extending u. Whitehead's Lemma (see for example Raghunathan's book [64]) implies then that u_g is cohomologicaly trivial, and in particular u is, a contradiction with our assumption on u. We conclude that L is not injective and thus there exists a non-trivial pure translation $(\mathrm{id}, u) \in \mathsf{X}$. Since $(g, u_g)(\mathrm{id}, u)(g, u_g)^{-1} = (\mathrm{id}, \phi(g)u)$, the Proposition will be proved once we have stablished that the additive group spanned by $\phi(\mathsf{G})u$ is V.

Lemma 2.13 provides a $g \in \mathsf{G}$ such that for all $\chi \in \Pi_\phi$ one has $(\phi(g)u)_\chi \neq 0$, where, for a vector $w \in V$, we have denoted by w_χ its componenent in the restricted weight space V^χ, following the restricted weight decomposition of V.

Since the additive group spanned by $\phi(\mathsf{G})u$ and that of $\phi(\mathsf{G})\phi(g)u$ coincide, we assume from now that the pure translation $(\mathrm{id}, u) \in \mathsf{X}$ is such that for all $\chi \in \Pi_\phi$ one has

$$u_\chi \neq 0.$$

Since Π_ϕ is finite we can consider $z \in \mathfrak{a}$ such that the values $\chi(z)$, for $\chi \in \Pi_\phi$, are pairwise distinct. In particular $\chi(z) \neq 0$ for any non-vanishing χ.

Let us fix a weight $\mu \in \Pi_\phi \setminus \{0\}$ and consider the linear map

$$\mathrm{R}^\mu = (\phi(\exp(z)) - \mathrm{id}) \prod_{\chi \in \Pi_\phi - \{0,\mu\}} \Big(2\phi(\exp(z)) - \phi\Big(\exp\Big(\big(1 + \frac{\log 2}{\chi(z)}\big) z \Big)\Big)\Big) \in \mathfrak{gl}(V).$$

One readily sees that

- $\mathrm{R}^\mu u$ belongs to the additive group spanned by $\phi(\mathsf{G})u$,
- the order on the above product is irrelevant as we are only considering elements of $\exp(\mathfrak{a})$, which commute; whence $\mathrm{R}^\mu u_\chi = 0$ for all $\chi \neq \mu$.

In particular,

$$\begin{aligned} \mathrm{R}^\mu u &= \mathrm{R}^\mu(u_\mu) \\ &= \Big((e^{\mu(z)} - 1) \prod_{\chi \in \Pi_\phi - \{0,\mu\}} \big(2e^{\mu(z)} - e^{\mu(z) + \log 2 \frac{\mu(z)}{\chi(z)}} \big) \Big)(u_\mu) \\ &= c \cdot u_\mu \neq 0, \end{aligned}$$

as the coefficient c is non-zero since $\chi(z) \neq \mu(z)$ for all $\chi \neq \mu$.

One concludes that, for all $a \in \mathfrak{a}$ one has $\phi(\exp(a))\mathrm{R}^\mu u = \exp(\mu(a)) c u_\mu$, so by also considering differences, we get that the line $\mathbb{R}(u_\mu)$ is contained in the additive group spanned by $\phi(\mathsf{G})u$. Now, the additive group spanned by $\phi(\mathsf{G})\mathbb{R}(u_\mu)$ coincides with the vector space spanned by $\phi(\mathsf{G})(u_\mu)$, which coincides with V by irreducibility of the representation ϕ, as desired. □

Lemma 6.2. *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ and $\psi : \mathsf{G} \to \mathsf{SL}(W)$ be two representations with ϕ strongly irreducible and such that, there exists $0 \neq \mu \in \Pi_\phi \setminus \Pi_\psi$. Let $\Gamma < \mathsf{G}$ be a Zariski-dense semigroup, $\mathsf{u}_V : \Gamma \to V$ a non-coboundary cocycle and $\mathsf{u}_W : \Gamma \to W$ a cocycle. Denote by*

$$\mathsf{u} = \mathsf{u}_V + \mathsf{u}_W : \Gamma \to V \oplus W.$$

Then the Zariski closure X *of* Γ_{u} *contains* $\mathsf{G} \ltimes V$*. If moreover* Γ_{u_W} *is Zariski-dense in* $\mathsf{G} \ltimes W$*, then* $\mathsf{X} = \mathsf{G} \ltimes (V \oplus W)$*.*

Proof. Let X be the Zariski closure of Γ_{u}. Consider the projection $\pi : \mathsf{G} \ltimes (V \oplus W) \to \mathsf{G} \ltimes V$ given by $\pi(g, v + w) = (g, v)$. Since $[\mathsf{u}]_V \neq 0$, Proposition 6.1 applies to give that the restriction of π to X is surjective. This in turn implies that for every $v \in V$ there exists $w \in W$ such that $(\mathrm{id}, v + w) \in \mathsf{X}$. Fix $\mu \in \Pi_{\phi} \setminus \Pi_{\psi}$ and choose v so that $v_\mu \neq 0$.

We consider now the restricted weight space decomposition

$$V \oplus W = \sum_{\chi \in \Pi_{\phi} \cup \Pi_{\psi}} (V \oplus W)^{\chi},$$

and denote, for $u \in V \oplus W$ by $u_\chi \in (V \oplus W)^{\chi}$ the associated component. Observe that $(V \oplus W)^{\mu} = V^{\mu}$ and thus $(v + w)_\mu = v_\mu$.

We now proceed again as in the proof of Proposition 6.1 by considering the (modified) operator

$$\mathrm{R}^{\mu} = (\phi(\exp(z)) - \mathrm{id}) \prod_{\chi \in \Pi_{\phi} \cup \Pi_{\psi} - \{0, \mu\}} \left(2\phi(\exp(z)) - \phi\Big(\exp\Big(\big(1 + \frac{\log 2}{\chi(z)}\big) z\Big)\Big) \right),$$

and applying to the vector $u = v + w$. The same arguments lead now to the desired inclusion.

To prove the last item, we consider the projection $\pi^{W} : \mathsf{G} \ltimes (V \oplus W) \to \mathsf{G} \ltimes W$ given by $(g, v + w) \mapsto (g, w)$. By assumption $\pi^{W}(\mathsf{X}) = \mathsf{G} \ltimes W$ whence for each $w \in W$ and $g \in \mathsf{G}$ there exists $v \in V$ with $(g, v + w) \in \mathsf{X}$. However, as we have established that $\mathsf{G} \ltimes V \subset \mathsf{X}$, we obtain that $(g, w) = (g, v + w) \cdot (\mathrm{id}, -v) \in \mathsf{X}$. Thus, X contains both $\mathsf{G} \ltimes V$ and $\mathsf{G} \ltimes W$, giving the result. □

Let us introduce the following definition.

Definition 6.3. A finite collection of strongly irreducible representations $\{\phi_i : \mathsf{G} \to V_i\}_{i \in I}$ is *disjoined* if we can order $I = [\![1, k]\!]$ such that ϕ_1 is non-trivial and for each $i \geq 2$ there exists $0 \neq \mu_i \in \Pi_{\phi_i} \setminus \bigcup_{l=1}^{i-1} \Pi_{\phi_l}$. A representation ϕ is *disjoined* if the collection of its factors is.

Remark 6.4. Observe that the Adjoint representation is always disjoined (regardless that G has isomorphic factors) since the restricted weights of each irreducible factor of this representation lie on different factors of $\mathfrak{a}^*$.

Corollary 6.5. *Let* $\{\phi_i : \mathsf{G} \to \mathsf{SL}(V_i)\}_1^k$ *be a disjoined collection. Let* $\Gamma < \mathsf{G}$ *be a Zariski-dense semigroup and consider for each* i *a non-coboundary cocycle* $\mathsf{u}_i : \Gamma \to V_i$*, and define* $\mathsf{u} = \sum_i \mathsf{u}_i : \Gamma \to \bigoplus_i V_i$*. Then* Γ_{u} *is Zariski-dense in* $\mathsf{G} \ltimes \left(\bigoplus_1^k V_i\right)$*.*

Proof. Follows by induction, Proposition 6.1 gives the base step, and the inductive step is given by Lemma 6.2. □

We fix from now on a disjoined representation $\phi : \mathsf{G} \to V$ and a cocycle $\mathsf{u} : \Gamma \to V$ over ϕ. Remark 3.3 implies that if $\dot{g} \in \Gamma$ is a (ϕ, X_0)-compatible element then $\mathsf{m}(\dot{g}, \mathsf{u}(\dot{g}))$ only depends on the class $[\mathsf{u}] \in H^1_{\phi}(\Gamma, V)$, so we consider the map $\mathsf{m} : \Gamma^{\phi} \times H^1_{\phi}(\Gamma, V) \to \mathcal{T}$ defined by

$$\mathsf{m}_{\mathsf{u}}(\dot{g}) = \mathsf{m}^{\dot{g}}(\mathsf{u}) := \mathsf{m}(\dot{g}, \mathsf{u}(\dot{g})),$$

where we have denoted by $\Gamma^{\phi} = \{(\phi, X_0)\text{-compatible elements of } \Gamma\}$.

By the very definition of m, the map is linear on the second variable and m_{u} identically vanishes when u is a co-boundary. So from Corollary 4.6 we conclude the following:

Corollary 6.6. *Let $\Gamma \subset \mathsf{G}$ be a Zariski-dense sub-semi-group. Let $\{\phi_i : \mathsf{G} \to \mathsf{SL}(V_i)\}_{i\in I}$ be a disjoined collection and for each i let $\mathsf{u}_i : \Gamma \to V_i$ be a cocycle over ϕ_i. Let $\phi = \bigoplus_i \phi_i$, and assume there exists, for each i, a (ϕ, X_0)-compatible $\dot{g}_i \in \Gamma$ such that $\mathsf{m}(\dot{g}_i, \mathsf{u}_i(\dot{g}_i)) \neq 0$. Denote by $V = \bigoplus_i V_i$ and by $\mathsf{u} = \sum_i \mathsf{u}_i : \Gamma \to V$. Then the affine limit cone $\mathscr{A}_{\Gamma_{\mathsf{u}}}$ has non-empty interior.*

Proof. By Remark 3.3, the fact that $\mathsf{m}(\dot{g}_i, \mathsf{u}_i(\dot{g}_i)) \neq 0$ implies that the class $[\mathsf{u}]_i \neq 0$. Corollary 6.5 then implies that Γ_{u} is Zariski-dense in $\mathsf{G} \ltimes_{\phi} V$ and thus the statement is reduced to Corollary 5.1. □

7. Compatible and θ-Anosov linear part, normalized Margulis spectra

Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be a disjoined representation and $\rho \in \mathfrak{A}_{\theta}(\Gamma, \mathsf{G})$. We say that ρ is (ϕ, X_0)-*compatible* if

$$\mathcal{L}_\rho \subset \mathfrak{X}_{\phi}.$$

A cocycle $\mathsf{u} \in H^1_{\phi\rho}(\Gamma, V)$ induces a $\mathfrak{T}$-valued translation cocycle $c : \Gamma \times \partial^2\Gamma \to \mathfrak{T}$ as in S. [67] (see also Ledrappier [53]), defined by

$$c\big(\gamma, (x, y)\big) = \pi^{\mathfrak{T}}\Big(\phi\big(\psi^{-1}_{(x,y)}\big)\pi^{X_\phi \cap Y_\phi, x_\phi \oplus y_\phi}\big(\mathsf{u}(\gamma^{-1})\big)\Big).$$

Indeed we have to check that for every pair $\gamma, h \in \Gamma$ one has

$$c\big(\gamma h, (x, y)\big) = c\big(h, (x, y)\big) + c\big(\gamma, h(x, y)\big),$$

which follows from a straightforward computation.

Applying S. [67, Proposition 3.1.1] we obtain a Hölder-continuous function

$$\mathcal{J}_{\mathsf{u}} : \mathsf{U}\Gamma \to \mathfrak{T}$$

such that for every hyperbolic $\gamma \in \Gamma$ one has $\ell_{[\gamma]}(\mathcal{J}_{\mathsf{u}}) = \mathsf{m}(\gamma, \mathsf{u}(\gamma))$. Thus, for every $\psi \in \operatorname{int}(\mathcal{L}_{\theta,\rho})^*$, the set of *normalized Margulis spectra*

$$\mathrm{MS}^{\psi}\big([\mathsf{u}]\big) = \overline{\left\{\frac{\mathsf{m}\big(\gamma, \mathsf{u}(\gamma)\big)}{\psi^{\gamma}(\rho)} : \gamma \in \Gamma_{\mathrm{h}}\right\}}$$

is convex and compact. Pairing the Ledrappier potential for u with an invariant probability measure should be thought of as a generalization of Goldman-Labourie-Margulis [31].

Remark 7.1. Under the assumptions of Corollary 6.6 for u, if Γ is replaced by $\rho(\Gamma)$ then, if w_0 acts trivially on $\mathfrak{T}$ then $\mathrm{MS}^{\psi}([\mathsf{u}])$ has non-empty interior.

The following will only be needed in Section 11.3. Recall from Smilga [71] that a representation is *non-swinging* if there exists $X_0 \in \mathrm{Fix}(\mathrm{i})$ such that $\chi(X_0) \neq 0$ for every $\chi \in \Pi_{\phi} - \{0\}$.

Proposition 7.2 (Kassel-Smilga [41])**.** *Let $\phi : \mathsf{G} \to \mathsf{SL}(V)$ be an irreducible, non-swinging representation such that $\mathfrak{T} = V^0 \neq 0$ and let Γ be a θ-Anosov (ϕ, X_0)-compatible subgroup of G. If $0 \in \operatorname{int}\mathrm{MS}^{\psi}([\mathsf{u}])$ then the action of $\rho(\Gamma)_{\mathsf{u}}$ on V is not* *proper. However, if* $\psi \in \operatorname{int}(\mathfrak{a}^+)^*$ *and* $0 \notin \mathrm{MS}^\psi([\mathsf{u}])$ *then the corresponding action is proper.*

We also have the following result by Kassel-Smilga [41] and Ghosh [27, Theorem 1.1.2] (who further requires that G is split and center free).

Proposition 7.3. *Let* $\phi : \mathsf{G} \to \mathsf{SL}(V)$ *be an irreducible non-swinging representation such that such that* $\mathfrak{T} = V^0 \neq 0$. *Let* $\rho : \Gamma \to \mathsf{G}$ *be a* θ*-Anosov* (ϕ, X_0)*-compatible subgroup of* G. *Then the action of* $\rho(\Gamma)_{\mathsf{u}}$ *on* V *is not proper if and only if there exists a sequence* $\{\gamma_n\} \subset \Gamma$ *such that* $\psi^{\gamma_n}(\rho) \to \infty$ *and such that* $\mathsf{m}(\gamma_n, \mathsf{u}(\gamma_n))$ *remains bounded, in particular* $0 \in \mathrm{MS}^\psi([\mathsf{u}])$.

In particular, if $0 \notin \mathrm{MS}^\psi([\mathsf{u}])$ then these Margulis space-times lie on the context of S. [67, § 3.4, 3.5, 3.6] and several results there apply directly.

Part 2. The cone of Jordan variations, normalizations, pressure

Throughout this Part we fix a faithful morphism $\rho : \Gamma \hookrightarrow \mathsf{G}$, so we often identify Γ with $\rho(\Gamma)$, for example we will say that $\gamma \in \Gamma$ is loxodromic if $\rho(\gamma)$ is. We will finally consider an integrable vector

$$v \in \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{G}).$$

For $\gamma \in \Gamma$ we denote by $\lambda^\gamma : \mathfrak{X}(\Gamma, \mathsf{G}) \to \mathfrak{a}$ the map $\lambda^\gamma(\eta) = \lambda(\eta(\gamma))$, and we let $\mathrm{d}\lambda^\gamma$ be its differential (when it is defined). Moreover, if $\varphi \in \mathfrak{a}^*$ then we let $\varphi^\gamma : \mathfrak{X}(\Gamma, \mathsf{G}) \to \mathbb{R}$ be the composition

$$\varphi^\gamma = \varphi \circ \lambda^\gamma : \rho \mapsto \varphi\big(\lambda\big(\rho(\gamma)\big)\big).$$

The vector $v \in \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{G})$ induces a cocycle $\mathsf{u}_v : \Gamma \to \mathfrak{g}$ defined by, for $\gamma \in \Gamma$,

$$\mathsf{u}_v(\gamma) = \frac{\partial}{\partial t}\Big|_{t=0} \rho_t(\gamma)\rho(\gamma)^{-1}.$$

The cocycle u_v is a co-boundary if and only if there exists $(s_t) \in \mathsf{G}$ with $s_0 = \mathrm{id}$ such that for all $\gamma \in \Gamma$ the curves $\rho_t(\gamma)$ and $s_t\rho(\gamma)s_t^{-1}$ have the same derivative. Equivalently, the curve ρ_t is tangent at 0 to the conjugacy class of ρ, which is also equivalent to the fact that the curve of characters $\rho_t \in \mathfrak{X}(\Gamma, \mathsf{G})$ has zero derivative.

We introduce two concepts which are the main object of this part.

Definition 7.4.

- The *cone of Jordan variations of* v is the cone associated to variations of Jordan projections:

$$\mathscr{V}\mathscr{J}_v := \overline{\Big\{\mathbb{R}_+ \cdot \mathrm{d}\lambda^\gamma(v) : \gamma \in \Gamma \text{ loxodromic}\Big\}} \subset \mathfrak{a}.$$

- Let $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$, then the *set of* ψ*-normalized variations* is

$$\mathbb{V}_v^\psi = \overline{\Big\{\frac{\mathrm{d}\lambda^\gamma(v)}{\psi^\gamma(\rho)} : \gamma \in \Gamma\Big\}} \subset \mathfrak{a}.$$

Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ on simple ideals and assume we've chosen the Cartan subspaces $\mathfrak{a}_i$ of $\mathfrak{g}_i$ so that $\mathfrak{a} = \bigoplus_i \mathfrak{a}_i$. Let $p_i : \mathfrak{g} \to \mathfrak{g}_i$ be the associated projections, choices have been made so that $p(\mathfrak{a}) = \mathfrak{a}_i$. Assume also the Weyl chambers $\mathfrak{a}_i^+$ where chosen so that $\mathfrak{a}^+ = \bigoplus_i \mathfrak{a}_i^+$.

The vector v has *full variation* if for every $i \in I$ the cocycle $p_i(\mathsf{u}_v)$ is not cohomologicaly trivial. It has moreover *full loxodromic variation* if for every i the cone $p_i(\mathscr{V}\mathscr{J}_v)$ is not $\{0\}$.

8. Variation of eigenvalues and some consequences of Part 1

We now apply Part 1 to a particular situation: we let $\phi = \mathrm{Ad} : \mathsf{G} \to \mathsf{SL}(\mathfrak{g})$ be the adjoint representation. The set of weights is the restricted root system Φ of $\mathfrak{g}$ and no root is w_0-invariant so the ideally neutral space

$$\mathcal{N} = \mathfrak{g}^0 = \mathfrak{m} \oplus \mathfrak{a}.$$

Writing $\mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m})$ one has that M preserves each factor $\mathfrak{g}^0 = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ and acts trivially on $\mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ so the neutralizing space is

$$\mathcal{T} = \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}.$$

Moreover, picking any $X_0 \in \mathfrak{a}^+ \cap \mathrm{Fix}(\mathrm{i})$ one readily sees that the (Ad, X_0)-compatible cone is nothing but the whole Weyl chamber $\mathfrak{a}^+$. The Margulis invariant is thus well defined for any $\mathsf{g} \in \mathsf{G} \ltimes \mathfrak{g}$ with loxodromic $\dot{\mathsf{g}} \in \mathsf{G}$ and one has

$$\mathsf{m}(\mathsf{g}) \in \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}.$$

Finally, observe that Ad is a disjoined representation (recall Definition 6.3).

8.1. The variation of the Kostant-Lyapunov-Jordan projection.

Let $(g_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathsf{G}$ be a differentiable curve with loxodromic $g = g_0$. We denote by $\vec{g} \in \mathsf{T}_{g_0}\mathsf{G}$ its derivative. Consider the affine transformation $\mathsf{g} \in \mathsf{G} \ltimes \mathfrak{g}$ whose linear part is g and translation vector

$$d_g L_{g^{-1}}(\vec{g}) = \frac{\partial}{\partial t}\Big|_{t=0} g_t g^{-1} \in \mathfrak{g}.$$

Then one has the following.

Proposition 8.1. *The* $\mathfrak{a}$*-factor of* $\mathsf{m}(\mathsf{g})$ *is* $\frac{\partial}{\partial t}\Big|_{t=0} \lambda(g_t)$.

The proof requires the following Lemma.

Lemma 8.2. *Consider a differentiable curve* $(s_t)_{(-\varepsilon,\varepsilon)} \subset \mathsf{G}$ *with* $s_0 = \mathrm{id}$ *and let* $h_t = s_t^{-1} g_t s_t$ *and* $\mathsf{h} \in \mathsf{G} \ltimes \mathfrak{g}$ *be defined as above, then* $\mathsf{m}(\mathsf{h}) = \mathsf{m}(\mathsf{g})$.

Proof. Indeed, explicit computation yields that the translation part of h is

$$\frac{\partial}{\partial t}\Big|_{t=0} h_t h^{-1} = \frac{\partial}{\partial t}\Big|_{t=0} g_t g^{-1} + \mathrm{Ad}(g_0)(\vec{s}) - \vec{s},$$

so the lemma follows by Remark 3.12. □

Proof of Proposition 8.1. Since g_0 is loxodromic, so is g_t for small enough t; denote by $x_t, y_t \in \mathcal{F}_\Delta$ the corresponding repelling and attracting flags. Since $\mathsf{m}(\mathsf{g})$ is invariant under conjugation, we can assume that $x_0 = [\check{\mathsf{P}}^\Delta]$ and $y_0 = [\mathsf{P}^\Delta]$, so that $g = g_0 = ma$ for some $m \in \mathsf{M}$ and $a \in \mathsf{A}$.

We can also consider a differentiable curve $s_t \in \mathsf{G}$ with $s_0 = \mathrm{id}$, that sends the pair (x_t, y_t) to $([\check{\mathsf{P}}^\Delta], [\mathsf{P}^\Delta])$, consequently by Lemma 8.2 one has $\mathsf{m}(\mathsf{h}) = \mathsf{m}(\mathsf{g})$, for $h_t := s_t^{-1} g_t s_t$.

We compute now $\mathsf{m}(\mathsf{h})$. Since h_t fixes $([\check{\mathsf{P}}^\Delta], [\mathsf{P}^\Delta])$ there exist $m_t \in \mathsf{M}$ and $a_t \in \mathsf{A}$ such that

$$h_t = m_t a_t$$

with $\lambda(g_t) = \lambda(a_t)$. Computing the derivative of $h_t h^{-1}$ one sees

$$\begin{aligned}\frac{\partial}{\partial t}\Big|_{t=0} h_t h^{-1} &= \frac{\partial}{\partial t}\Big|_{t=0} m_t a_t a^{-1} m^{-1} \\ &= \vec{m} m^{-1} + \vec{a} a^{-1},\end{aligned} \tag{8.1}$$

since M and A commute. The Margulis invariant of g is then computed by considering the eigenspace decomposition of $\mathrm{Ad}(ma)$, which is nothing but the root space decomposition

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a} \oplus \mathfrak{n} \oplus \check{\mathfrak{n}},$$

and projecting the vector (8.1) onto $\mathcal{T} = \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$ parallel to this decomposition. The $\mathfrak{a}$-factor of $\mathsf{m}(\mathsf{g})$ is then $\vec{a} a^{-1}$, as desired. □

The same proof above actually gives the following:

Corollary 8.3. *Let $\mathsf{G}_\mathbb{C}$ be a complex semi-simple algebraic group. Let $\mathfrak{a}_\mathbb{C}$ be a Cartan subalgebra of $\mathsf{G}_\mathbb{C}$ and let $\lambda_\mathbb{C} : \mathsf{G}_\mathbb{C} \to \exp(\mathfrak{a}_\mathbb{C})$ be the complex Jordan projection. Let $(g_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathsf{G}_\mathbb{C}$ be a differentiable curve with loxodromic g_0. Then*

$$\mathsf{m}(\mathsf{g}) = \frac{\partial}{\partial t}\Big|_{t=0} \lambda_\mathbb{C}(g_t) \in \mathfrak{a}_\mathbb{C}.$$

Proof. Considering $\mathsf{G}_\mathbb{C}$ as a real-algebraic group one has that $\mathfrak{m}$ is abelian, so $\mathfrak{m} = \mathfrak{Z}(\mathfrak{m})$ and $\mathfrak{a}_\mathbb{C} = \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$. In the course of the proof of Proposition 8.1 one may observe that, the $\mathfrak{Z}(\mathfrak{m})$-factor of $\mathsf{m}(\mathsf{g})$ is the projection of $\vec{m} m^{-1} \in \mathfrak{m} = [\mathfrak{m}, \mathfrak{m}] \oplus \mathfrak{Z}(\mathfrak{m})$ parallel to $[\mathfrak{m}, \mathfrak{m}]$, which readily gives the result. □

8.2. **Every functional sees eigenvalue variations.** We now prove Theorem A.

Corollary 8.4. *Assume $\rho(\Gamma)$ is Zariski-dense. If v has full loxodromic variation, then $\mathscr{V}\mathscr{J}_v$ is convex and has non-empty interior. In particular, if ρ is a regular point of $\mathfrak{X}(\Gamma, \mathsf{G})$ and $\varphi \in \mathfrak{a}^*$ does not annihilate any of the $\mathfrak{a}_i$'s, then the set $\{\mathrm{d}\varphi^\gamma : \gamma \in \Gamma\}$ spans the co-tangent space $\mathsf{T}^*_\rho \mathfrak{X}(\Gamma, \mathsf{G})$.*

Proof. Proposition 8.1 places the statement in the conditions of Corollary 6.6 where $V_i = \mathfrak{g}_i$. The Corollary applies since the Adjoint representation is disjoined (Remark 6.4). Thus, the affine limit cone

$$\mathscr{A}_{\rho(\Gamma)_{\mathfrak{u}_v}} \subset \mathfrak{Z}(\mathfrak{m}) \oplus \mathfrak{a}$$

is convex and has non-empty interior, whence it's projection onto the second factor also, giving the conclusion. □

We introduce for convenience the following definition.

Definition 8.5. If H is a reductive subgroup of G then an *adjoint factor of* H is a collection of irreducible factors of the representation $\mathrm{ad}_\mathfrak{g} |\mathfrak{h} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$. An adjoint factor is *disjoined* if the associated representation is.

If $\rho(\Gamma)$ has semi-simple Zariski closure H, then the cohomology $H^1_{\mathrm{Ad}}(\Gamma, \mathfrak{g})$ splits as

$$H^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g}) = \bigoplus_{V_\mathsf{H} \text{ irreducible adjoint factor}} H^1_{\mathrm{Ad}\,\rho}(\Gamma, V_\mathsf{H}).$$

Thus one obtains the following refinement of Corollary 8.4 whose proof is identical. Recall Definition 6.3 of a disjoined representation ϕ.

Corollary 8.6. *Assume $\rho(\Gamma)$ has semi-simple Zariski closure H and assume that $\mathcal{L}_\rho \cap \operatorname{int}\mathfrak{a}^+ \neq \emptyset$. Let V_{H} be a disjoined adjoint factor and $\phi = \mathrm{Ad}_{\mathsf{G}}(\mathsf{H})|V_{\mathsf{H}}$. Assume $\mathsf{u} \in H^1_{\phi\rho}(\Gamma, V_{\mathsf{H}}) \setminus \{0\}$ is integrable, then for every $\gamma \in \Gamma$ with loxodromic $\rho(\gamma)$ one has*

$$\mathrm{d}\lambda^\gamma(\mathsf{u}) \in V_{\mathsf{H}} \cap \mathfrak{a}.$$

Moreover, if $[\mathsf{u}]$ projects non-trivially to the twisted cohomology associated to each irreducible factor of ϕ, then $\mathscr{V}\mathscr{J}_v$ is convex and has non-empty interior in $V_{\mathsf{H}} \cap \mathfrak{a}$. Consequently, for every $\varphi \in \mathfrak{a}^*$ such that $V_{\mathsf{H}} \cap \mathfrak{a} \subsetneq \ker\varphi$ there exists $\gamma \in \Gamma$ such that

$$\mathrm{d}\varphi^\gamma(\mathsf{u}) \neq 0.$$

Proof. Since H contains a G-loxodromic element the 0-weight space of the representation $\phi : \mathsf{H} \to \mathsf{GL}(V_{\mathsf{H}})$ verifies

$$\mathfrak{a} \cap V_{\mathsf{H}} \subset (V_{\mathsf{H}})^0 \subset (\mathfrak{m} \oplus \mathfrak{a}) \cap V_{\mathsf{H}}.$$

It follows that the $\mathfrak{a}$-factor of the Margulis invariant of $(\eta(\gamma), \mathsf{u}(\gamma))$, as an element of $\mathsf{H} \ltimes_\phi V_{\mathsf{H}}$, coincides with $\mathrm{d}\lambda^\gamma(\mathsf{u})$, so non-empty interior of $\mathscr{V}\mathscr{J}_{\mathsf{u}}$ in $V_{\mathsf{H}} \cap \mathfrak{a}$ follows from Corollary 6.6. □

9. Zariski-density of elements with full variation

We further assume throughout this section that $\rho(\Gamma)$ is Zariski-dense and that v has full loxodromic variation.

Proposition 9.1. *Assume $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$ has full loxodromic variation and Zariski-dense base point. Then the set*

$$\Gamma_{\text{full}} = \{\textit{loxodromic and full variation } g \in \Gamma\}$$

is Zariski-dense in G. Moreover, the set $\{\lambda(g) : g \in \Gamma_{\text{full}}\}$ intersects every open cone contained in $\mathcal{L}_\rho$.

The proof follows the same lines as the analogous statement for loxodromic elements by Benoist [3] (see [5, Theorem 6.36]).

Lemma 9.2. *Let $\mathbb{K}$ be a field and consider $w, g, h \in \mathsf{GL}(d, \mathbb{K})$, then for every $N \in \mathbb{N}$ the Zariski closure of $\{wg^nh^n : n \in [\![N, \infty)\}$ contains the product wgh. Analogously, the Zariski closure of $\{g^nh^nw : n \in [\![N, \infty)\}$ contains ghw.*

Proof. Let $I = \{p \in \mathbb{R}[x_{ij}] : p(wg^nh^n) = 0 \ \forall n \geq N\}$ be the associated ideal. We must show that for every $p \in I$ it holds $p(wgh) = 0$.

Let us consider the map $T : \mathbb{R}[x_{ij}] \to \mathbb{R}[x_{ij}]$ defined, for $X = (x_{ij})_{ij}$, by

$$Tp(X) = p(wg^N w^{-1} X h^N).$$

It is an isomorphism that preserves I. Moreover, the finite-dimensional vector space

$$I^m = \{p \in I \text{ of total degree } \leq m\}$$

verifies $T(I^m) \subset I^m$ which yields, since T is invertible, that $T(I^m) = I^m$, and thus $T(I) = I$. Consider then $q \in I$ and let $p \in I$ be such that $Tp = q$, then $q(wgh) = p(wg^{N+1}h^{N+1}) = 0$, as desired. □

Lemma 9.3. *Let $g, h \in \Gamma$ be loxodromic and transverse, if g has full variation and $k \in \mathbb{N}$ is such that*

$$\forall i \ \ p_i\big(k\mathrm{d}\lambda^g(v) + \mathrm{d}\lambda^h(v)\big) \neq 0,$$

then for all large enough n (depending on k) the element $(g^k)^nh^n$ has full variation.

Proof. By Theorem 2.8 we have

$$\frac{\lambda\big((g^k)^n h^n\big)}{n} - \big(\lambda(g^k) + \lambda(h)\big) \xrightarrow{n\to\infty} 0$$

and the convergence is uniform about g^k and h. Since we're considering an analytic variation $u \mapsto \rho_u$ and λ is an analytic function when restricted to loxodromic elements of G, we can differentiate both sides in the convergence to obtain

$$\frac{\mathrm{d}\lambda^{g^{kn}h^n}(v)}{n} - k\mathrm{d}\lambda^g(v) - \mathrm{d}\lambda^h(v) \xrightarrow{n\to\infty} 0.$$

By assumption we have, for every i, $p_i(k\mathrm{d}\lambda^g(v) + \mathrm{d}\lambda^h(v)) \neq 0$ so the above convergence implies the lemma. ☐

Lemma 9.4. *Let $\gamma \in \Gamma$ be loxodromic, then there exists $g \in \Gamma_{\text{full}}$ transverse to γ.*

Proof. Consider a loxodromic $g \in \Gamma$ with full variation, the existence of such g is guaranteed by Theorem A. By means of Zariski density of $\rho(\Gamma)$, we can find a loxodromic h that is transverse to both γ and g. By Lemma 9.3 elements of the form $g^n h^m$, for arbitrary large n and m, have full variation, and analogously for $h^m g^n$. Again for large enough m, n, the pairs $h^m g^n$ and $g^n h^m$ are transverse, so we can find, by Lemma 9.3 again, large enough k, l so that

$$(h^m g^n)^k (g^n h^m)^l$$

has full variation. Moreover, the attracting flag of the latter element is arbitrarily close to h^+ and the repelling flag is close to h_-, thus $(h^m g^n)^k (g^n h^m)^l$ has full variation and is transverse to γ. ☐

Lemma 9.5. *Let g, h be loxodromic and transverse, assume moreover that $g \in \Gamma_{\text{full}}$ then, the products hg and gh belong to the Zariski closure $\overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$. Moreover, the semi-group spanned by $\{gh, g\}$ is also contained in $\overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$.*

Proof. Since g has full variation, there exist K such that for all $k \geq K$ one has

$$\forall i \; p_i\big(\mathrm{d}\lambda^{g^k}(v) + \mathrm{d}\lambda^h(v)\big) = p_i\big(k\mathrm{d}\lambda^g(v) + \mathrm{d}\lambda^h(v)\big) \neq 0.$$

For every such k, Lemma 9.3 implies that for all large enough n (depending on k) one has $(g^k)^n h^n \in \Gamma_{\text{full}}$. Lemma 9.2 implies that $g^k h \in \overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$, for all $k \geq K$ and thus Lemma 9.2 again gives $gh \in \overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$.

We now show that the semi-group spanned by $\{gh, g\}$ is also contained in $\overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$. To this end, consider an arbitrary word w on the letters gh and g, and assume by induction that $w \in \overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$. We will show that the words

$$ghw, wgh, gw, wg$$

are all contained in $\overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$. Since g and h are transverse and we are only considering positive powers in w, the word w is transverse to g, and to any element of the form $g^m h^n$ for positive $n, m \in \mathbb{N}$.

Since $g \in \Gamma_{\text{full}}$, then the first statement of this lemma implies that $wg \in \overline{\Gamma_{\text{full}}}^{\mathsf{Z}}$ as desired.

Moreover, again since $g \in \Gamma_{\text{full}}$, the paragraph above establishes that, for all $k \geq K$ and all $n \geq N(k)$, one has $(g^k)^n h^n \in \Gamma_{\text{full}}$. Thus, since w is transverse to $(g^k)^n h^n$, the first item of this Lemma gives that $\forall k \geq K$, $\forall n \geq N(k)$

$$w(g^k)^n h^n \in \overline{\Gamma_{\text{full}}}^{\mathsf{Z}},$$

which implies, by applying twice Lemma 9.2 that $wgh \in \overline{\Gamma_{\text{full}}}^{\mathrm{Z}}$ as desired. The other inclusions are analogous. □

Proof of Proposition 9.1. We will show that the Zariski closure of Γ_{full} contains all loxodromic elements of $\rho(\Gamma)$, which are in turn Zariski-dense by Benoist [3] (see for example [5, Theorem 6.36]).

Consider then $\gamma \in \Gamma$ loxodromic. By Lemma 9.4 there exists $g \in \Gamma_{\text{full}}$ transverse to γ. Lemma 9.5 establishes that the semigroup spanned by $\{g\gamma, g\}$ is contained in $\overline{\Gamma_{\text{full}}}^{\mathrm{Z}}$, but the Zariski closure of a semi-group is a group (c.f. [5, Lemma 6.15]) thus the group spanned by $\{g\gamma, g\}$ is contained in $\overline{\Gamma_{\text{full}}}^{\mathrm{Z}}$ and in particular so does $\gamma = g^{-1}(g\gamma)$, as desired.

We finally establish the last statement in the Proposition. Consider an open cone $\mathscr{C} \subset \mathcal{L}_\rho$, $\gamma \in \Gamma$ loxodromic with $\lambda(\gamma) \in \mathscr{C}$ and $g \in \Gamma_{\text{full}}$ transverse to γ. Consider then $t \in \mathbb{R}_+ \setminus \mathbb{Q}$ so that $t\lambda(\gamma) + \lambda(g) \in \mathscr{C}$, by the abundance of such t's we may further assume that for all i $p_i\big(t\mathrm{d}\lambda^\gamma(v) + \mathrm{d}\lambda^g(v)\big) \neq 0$. Consider then a sequence of rationals in lowest terms $m_n/q_n \to t$, since $t \in \mathbb{R} \setminus \mathbb{Q}$ we have $\min\{m_n, q_n\} \to \infty$. Lemma 2.9 implies then that

$$\lim_{n\to\infty} \frac{\lambda(\gamma^{m_n} g^{q_n})}{q_n} = t\lambda(\gamma) + \lambda(g) \in \mathscr{C},$$

so $\lambda(\gamma^{m_n} g^{q_n}) \in \mathscr{C}$ for big enough n. Moreover, again by analyticity of our curve and uniform convergence on the above limit, we get by differentiating both sided of the limit that $\gamma^{m_n} g^{q_n} \in \Gamma_{\text{full}}$ for all large enough n. □

In fact, the above proof with minor modifications gives the following stronger result.

Remark 9.6. Assume $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$ has full loxodromic variation and Zariski-dense base point and fix a finite collection of hyperplanes $\mathfrak{U}$ of $\mathfrak{a}$. Then the set

$$\Gamma_{\mathfrak{U}} = \{\text{loxodromic } g \in \Gamma : \ \mathrm{d}\lambda^\gamma(v) \notin \bigcup_{U \in \mathfrak{U}} U\}$$

is Zariski-dense in G. Moreover, the set $\{\lambda(g) : g \in \Gamma_{\mathfrak{U}}\}$ intersects every open cone contained in $\mathcal{L}_\rho$.

9.1. **Convexity of normalized variations.**

Proposition 9.7. *Let $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$ have full loxodromic variation and Zariski-dense base-point then, the set of normalized variations is convex.*

Proof. Consider $\gamma, h \in \Gamma_{\text{full}}$. We first assume that γ and h are transversally loxodromic. In this case, using the argument from the proof of Proposition 9.1, we obtain that for every irrational $t \in \mathbb{R}_+$ one has $\lambda(\gamma^{m_n} h^{q_n})/q_n \to t\lambda(\gamma) + \lambda(h)$ which in turn gives:

$$\frac{\psi^{\gamma^{m_n} h^{q_n}}(\rho)}{q_n} \xrightarrow[n\to\infty]{} t\psi^\gamma(\rho) + \psi^h(\rho),$$
$$\frac{\mathrm{d}\lambda^{\gamma^{m_n} h^{q_n}}(v)}{q_n} \xrightarrow[n\to\infty]{} t\mathrm{d}\lambda^\gamma(v) + \mathrm{d}\lambda^h(v).$$

Combining both equations and since $\mathbb{V}_v^\psi$ is closed, we obtain that for every $t \in \mathbb{R}_+$

$$\frac{t\mathrm{d}\lambda^\gamma(v) + \mathrm{d}\lambda^h(v)}{t\psi^\gamma(\rho) + \psi^h(\rho)} \in \mathbb{V}_v^\psi.$$

So letting now $t = \psi^h(\rho)/\psi^\gamma(\rho)$ we obtain $\frac{1}{2}\left(\frac{\mathrm{d}\lambda^\gamma(v)}{\psi^\gamma(\rho)} + \frac{\mathrm{d}\lambda^h(v)}{\psi^h(\rho)}\right) \in \mathbb{V}_v^\psi$, as desired.

If γ and h are not transversally proximal then we replace h by some element of the form fh^nq for big enough n and well chosen $f, q \in \Gamma$ whose existence is guaranteed by Zariski-density of Γ. □

10. Theorem B: Base-point independence

We introduce, for convenience, the following definitions.

Definition 10.1.

- The *support* of $\varphi \in \mathfrak{a}^*$ is $\operatorname{supp}\varphi = \{\sigma \in \Delta : \langle \varphi, \sigma \rangle \neq 0\}$. Equivalently, upon writing $\varphi = \sum_{\sigma \in \Delta} \varphi_\sigma \varpi_\sigma$, one has $\sigma \in \operatorname{supp}\varphi$ if and only if $\varphi_\sigma \neq 0$. In particular, $\varphi \in (\mathfrak{a}_\vartheta)^*$ if and only if $\operatorname{supp}\varphi \subset \vartheta$.
- If $g \in \mathsf{G}$ we say that α *strictly minimizes* g *among* $\operatorname{supp}\varphi$ if $\alpha \in \operatorname{supp}\varphi$ and

$$\alpha(\lambda(g)) < \sigma(\lambda(g)) \ \forall \sigma \in \operatorname{supp}\varphi - \{\alpha\}. \tag{10.1}$$

The main result of this section is the following.

Theorem 10.2. *Let $\rho : \Gamma \hookrightarrow \mathsf{G}$ be a Zariski-dense sub-semigroup and $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$ have full loxodromic variation. Consider $\varphi \in \mathfrak{a}^*$ and assume there exist $\alpha \in \operatorname{supp}\varphi$ with $\dim \mathfrak{g}_\alpha = 1$ and $g \in \Gamma$ such that α strictly minimizes g among $\operatorname{supp}\varphi$. Then given $\psi \in \mathfrak{a}^*$ there exists a loxodromic $\gamma \in \Gamma$ such that $\mathrm{d}\varphi^\gamma(v) - \psi^\gamma(\rho) \notin \mathbb{Z}$.*

Corollary 10.3. *In the assumptions of Theorem 10.2, the additive group spanned by the pairs $\{(\mathrm{d}\varphi^\gamma(v), \lambda^\gamma(\rho)) : \gamma \in \Gamma \text{ loxodromic}\}$ is dense in $\mathbb{R} \times \mathfrak{a}$.*

Proof. Otherwise, there exist $(a, \psi) \in \mathbb{R} \times \mathfrak{a}^*$ s.t. for all loxodromic $\gamma \in \Gamma$ it holds $a\varphi(\mathrm{d}\lambda^\gamma(v)) + \psi(\lambda^\gamma(\rho)) \in \mathbb{Z}$. If $a \neq 0$ then $a\varphi$ verifies the assumptions of Theorem 10.2 giving a contradiction. If $a = 0$ this is contained in Benoist's Theorem 2.14. □

Example 10.4. For $g \in \mathsf{PSL}(2, \mathbb{C})$ denote by $|\gamma|$ its translation length on the hyperbolic 3-space $\mathbb{H}^3$. Let $\mathcal{T}(S)$ be the Teichmüller space of S as above. Consider then a Zariski-dense quasi-Fuchsian representation $\eta : \pi_1 S \to \mathsf{PSL}(2, \mathbb{C})$ and a non-zero $v = \vec{\rho} \in \mathsf{T}_\rho\mathcal{T}(S)$. Assume there exists $g \in \pi_1 S$ with $|\rho g| < |\eta g|$. Then by Corollary 10.3 the group spanned by the pairs

$$\Big\{ \big((\partial/\partial t)_{t=0} |\rho_t \gamma|, |\eta\gamma| \big) : \gamma \in \pi_1 S \Big\}$$

is dense in $\mathbb{R}^2$.

Recall from § 2.6 that we have a projection $\pi_\vartheta : \mathfrak{a} \to \mathfrak{a}_\vartheta$.

Corollary 10.5. *Let $\vartheta \subset \Delta$ be such that $\dim \mathfrak{g}_\sigma = 1$ for all $\sigma \in \vartheta$. Let $\rho : \Gamma \hookrightarrow \mathsf{G}$ be a Zariski-dense sub-semi-group and consider an integrable, full loxodromic variation $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma, \mathsf{G})$. Then the group spanned by*

$$\big\{ \big(\mathrm{d}\lambda_\vartheta^\gamma(v), \lambda^\gamma(\rho)\big) : \gamma \in \Gamma \text{ loxodromic} \big\}$$

is dense in $\mathfrak{a}_\vartheta \times \mathfrak{a}$. In particular, for any $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$ the set $\mathbb{V}_{\vartheta,v}^\psi$ has non-empty interior.

Proof. Otherwise there exist $\varphi \in (\mathfrak{a}_\vartheta)^*$ and $\psi \in \mathfrak{a}^*$ such that for all loxodromic $\gamma \in \Gamma$ one has $\mathrm{d}\varphi^\gamma(v) - \psi^\gamma(\rho) \in \mathbb{Z}$. Since $\rho(\Gamma)$ is Zariski-dense its limit cone has non-empty interior, there exists then $g \in \Gamma$ such that values $\sigma^g(\rho)$ for $\sigma \in \vartheta$ are pairwise distinct. Since $\operatorname{supp}\varphi \subset \vartheta$ and all roots in ϑ have 1-dimensional root space, there exists $\alpha \in \operatorname{supp}\varphi$ with 1-dimensional root-space that strictly minimizes g among $\operatorname{supp}\varphi$. This contradicts Theorem 10.2. The last statement now follows since by Proposition 9.7 $\mathbb{V}^{\psi}_{\vartheta,v}$ is convex. □

10.1. **Strongly transversally proximal elements.** We recall some notation of § 2.10. Let V be a finite dimensional real vector space and $g \in \operatorname{End}(V)$ be proximal, with top eigenvalue (in modulus) denoted by $\mu_1(g)$. We consider $\beta_g \in V^*$ and $v_g \in g^+$ such that $\ker \beta_g = g^-$ and $\beta_g(v_g) = 1$, we also let

$$\pi_g(w) = \beta_g(w)v_g.$$

We finally let $V_2(g)$ be the generalized eigenspace of g associated to $\mu_2(g)$, the second (in modulus) eigenvalue and τ_g be the only projection over $V_2(g)$ whose kernel is g-invariant.

Recall from Definition 2.7 that if $h \in \operatorname{End}(V)$ is also proximal, then g and h are *transversally proximal* if

$$\beta_g(v_h)\beta_h(v_g) \neq 0,$$

and *strongly transversally proximal* if $\beta_h(\tau_g v_h) \neq 0$. Recall that an element of $\operatorname{End}(V)$ is *semi-simple* if it is diagonalizable over $\mathbb{C}$. Finally, recall Equation (2.20) defining the multiplicative cross ratio B_1 of two decomposition of V into a line and a hyperplane.

Lemma 10.6 (Benoist-Quint [5, Lemma 7.15])**.** *Let $g, h \in \operatorname{End}(V)$ be transversally proximal, then for m, n big enough $g^n h^m$ is proximal and*

$$c_n(g,h) := \lim_{m\to\infty} \operatorname{Trace}(\pi_g \pi_{g^n h^m}) = \frac{\operatorname{Trace}(\pi_g g^n \pi_h)}{\operatorname{Trace}(g^n \pi_h)} = \mathsf{B}_1(g^+, g^-, g^n h^+, h^-). \tag{10.2}$$

If g and h are moreover strongly transversally proximal and g is semi-simple, then the sequence

$$\Big(\frac{\mu_1(g)}{\mu_2(g)}\Big)^n \log|c_n(g,h)|$$

is bounded. Moreover, let g_e be the elliptic component of g in Jordan's decomposition and n_k a sequence such that $g_e^{n_k}|V_2(g) \xrightarrow[k\to\infty]{} \operatorname{id}|V_2(g)$. Then,

$$\lim_{k\to\infty} \Big(\frac{\mu_1(g)}{\mu_2(g)}\Big)^{n_k} \log|c_{n_k}(g,h)| = \frac{-\beta_h(\tau_g v_h)}{\mathsf{B}_1(g^+, g^-, h^+, h^-)} \neq 0, \tag{10.3}$$

and the convergence is moreover uniform on h.

Proof. We focus on the second statement which is slightly different from what is found in [5]. Using Equation (10.2) we compute

$$\log|c_n(g,h)| \underset{n\to\infty}{\sim} c_n(g,h) - 1 = \frac{\operatorname{Trace}\big((\pi_g - 1)g^n \pi_h\big)}{\operatorname{Trace}(g^n \pi_h)}.$$

The denominator is easily controlled, indeed

$$\frac{\operatorname{Trace}(g^n \pi_h)}{\mu_1(g)^n} = \frac{\beta_h(g^n v_h)}{\mu_1(g)^n} \xrightarrow[n\to\infty]{} \beta_h\big(\pi_g(v_h)\big) = \mathsf{B}_1(g^+, g^-, h^+, h^-). \tag{10.4}$$

We now study the numerator. Observe that $\operatorname{Trace}\big((1-\pi_g)g^n\pi_h\big) = \beta_h(g^n\tau_g v_h) + o(\mu_2(g)^n)$. Since g is semi-simple, $V_2(g)$ decomposes as $\bigoplus_i W_i$ where each W_i is g-invariant and $g|W_i = \mu_2(g)K_i$, where $K_i : W_i \to W_i$ lies in an abelian compact group. Thus, the sequence

$$\frac{\beta_h(g^n\tau_g v_h)+o(\mu_2(g)^n)}{\mu_2(g)^n}$$

is bounded.

Moreover, considering the sequence $n_k\to\infty$ as in the statement, for all i one has $K_i^{n_k}\to \operatorname{id}$. Since g is semi-simple we deduce that

$$\frac{g^{n_k}|V_2(g)}{\mu_2(g)^{n_k}}\xrightarrow[k\to\infty]{}\operatorname{id},$$

so one concludes

$$\frac{\operatorname{Trace}\big((1-\pi_g)g^{n_k}\pi_h\big)}{\mu_2(g)^{n_k}}=\frac{\beta_h(g^{n_k}\tau_g v_h)}{\mu_2(g)^{n_k}}+\frac{o(\mu_2(g)^{n_k})}{\mu_2(g)^{n_k}}\xrightarrow[k\to\infty]{}\beta_h(\tau_g v_h),$$

as desired. ☐

If we suppose now that $g,h\in\mathsf{G}$ are ϑ-proximal, then we say they are *strongly transversally ϑ-proximal* if for every $\sigma\in\vartheta$ the maps $\phi_\sigma g$ and $\phi_\sigma h$ are strongly transversally proximal. For such a pair and $n\in\mathbb{N}$ we define the vector

$$\nu_n^\vartheta(g,h)\in\mathfrak{a}_\vartheta\ \text{ so that }\ \forall\sigma\in\vartheta,\ \varpi_\sigma\big(\nu_n^\vartheta(g,h)\big):=\log\big|c_n\big(\phi_\sigma g,\phi_\sigma h\big)\big|. \tag{10.5}$$

Proposition 10.7. *Let $g\in\mathsf{G}$ be loxodromic and consider $h\in\mathsf{G}$ so that g and h are strongly transversally ϑ-proximal. Consider $\varphi\in(\mathfrak{a}_\vartheta)^*$ and let $\alpha\in\operatorname{supp}\varphi$ be such that*

$$\alpha(\lambda(g))=\min\{\sigma(\lambda(g)):\sigma\in\operatorname{supp}\varphi\}. \tag{10.6}$$

If α has multiplicity 1 and is the only root in $\operatorname{supp}\varphi$ realizing the above minimum then there exists $\kappa^\varphi(g,h)\in\mathbb{R}$ such that

$$\varphi\big(\nu_{2n}^\vartheta(g,h)\big)e^{2n\alpha(\lambda(g))}\xrightarrow[n\to\infty]{}\kappa^\varphi(g,h)\neq 0. \tag{10.7}$$

In the latter case, the map $(\gamma,\eta)\mapsto\kappa^\varphi(\gamma,\eta)$ is analytic on both variables on a neighborhood of $(g,h)\in\mathsf{G}^2$ and the above convergence is uniform on a neighborhood of g and h.

Proof. By definition of $\nu_n^\vartheta(g,h)$, upon writing $\varphi=\sum_{\sigma\in\vartheta}\varphi_\sigma\varpi_\sigma$ one has

$$\begin{aligned}\varphi\big(\nu_n^\vartheta(g,h)\big)&=\sum_{\sigma\in\vartheta}\varphi_\sigma\varpi_\sigma\big(\nu_n^\vartheta(g,h)\big)\\&=\sum_{\sigma\in\vartheta}\varphi_\sigma\log\big|c_n(\phi_\sigma g,\phi_\sigma h)\big|.\end{aligned}$$

Considering, for each $\sigma\in\vartheta$, the fundamental representation associated to ϖ_σ we see by Lemma 10.6 that $e^{n\sigma(\lambda(g))}\varpi_\sigma\big(\nu_n^\vartheta(g,h)\big)$ is bounded, thus if α is the only root realizing the minimum in Equation (10.6) we have, for every $\sigma\in\operatorname{supp}\varphi-\{\alpha\}$ that

$$e^{n\alpha(\lambda(g))}\varpi_\sigma\big(\nu_n^\vartheta(g,h)\big)\xrightarrow[n\to\infty]{}0.$$

Moreover, since $\dim\mathfrak{g}_\alpha=1$, $V_2(\phi_\alpha(g))$ is 1-dimensional. Thus

$$\Big(\frac{\mu_1(\phi_\alpha(g))}{\mu_2(\phi_\alpha(g))}\Big)^2=e^{2\alpha(\lambda(g))}$$

and combining with the last statement of Lemma 10.6 we obtain

$$\varphi\big(\nu^{\vartheta}_{2n}(g,h)\big)e^{2n\alpha(\lambda(g))}\xrightarrow[n\to\infty]{}\varphi_\alpha\frac{-\beta_{\phi_\alpha h}(\tau_{\phi_\alpha g}v_{\phi_\alpha h})}{\varpi_\alpha(\mathfrak{C}_\vartheta(g^+,g^-,h^+,h^-))}:=\kappa^{\varphi}(g,h). \tag{10.8}$$

The convergence is uniform on a neighborhood of h so we now treat uniform convergence in g. If g' is close to g then, by one-dimensionality of $\mathfrak{g}_\alpha$, g' also acts as a homothety with ratio $\mu_2(\phi_\alpha g')$ on $V_2(\phi_\alpha g')$ and thus uniform convergence follows. Analyticity of $\kappa^{\varphi}(\cdot,h)$ follows as, since g is loxodromic, the map $\tau_{\phi_\alpha g}$ varies analytically about g. □

For a real-analytic curve $\big(t\mapsto(g_t,h_t)\big)_{t\in(-\varepsilon,\varepsilon)}$ with $g=g_0$ and $h=h_0$ strongly transversally ϑ-proximal, with g loxodromic, we denote for every $n\in\mathbb{N}$ and $\varphi\in(\mathfrak{a}_\vartheta)^*$ by $\nu^{(g,h)}_n(t):=\nu_n(g_t,h_t)$ and $\kappa^{\varphi,(g,h)}(t)=\kappa^{\varphi}(g_t,h_t)$. We also let

$$\dot\nu^{(g,h)}_n=\frac{\partial}{\partial t}\Big|_{t=0}\nu^{(g,h)}_n(t)\quad\text{and}\quad\dot\kappa^{\varphi,(g,h)}=\frac{\partial}{\partial t}\Big|_{t=0}\kappa^{\varphi,(g,h)}(t).$$

Corollary 10.8. *Consider a real-analytic curve $t\mapsto(g_t,h_t)$ for $t\in(-\varepsilon,\varepsilon)$ with $g=g_0$ and $h=h_0$ strongly transversally ϑ-proximal, and assume g is loxodromic. Consider $\varphi\in(\mathfrak{a}_\vartheta)^*$ and assume there exists $\alpha\in\operatorname{supp}\varphi$ with $\dim\mathfrak{g}_\alpha=1$ and so that*

$$\alpha(\lambda(g))<\sigma(\lambda(g))\ \forall\sigma\in\operatorname{supp}\varphi-\{\alpha\}. \tag{10.9}$$

Then,

$$\varphi\big(\dot\nu^{(g,h)}_{2n}\big)e^{2n\alpha(\lambda(g))}+\varphi\big(\nu^{(g,h)}_{2n}\big)2n\Big(\frac{\partial}{\partial t}\Big|_{t=0}\alpha\big(\lambda(g_t)\big)\Big)e^{2n\alpha(\lambda(g))}\xrightarrow[n\to\infty]{}\dot\kappa^{\varphi,(g,h)}.$$

Proof. Assumptions are made so that Proposition 10.7 applies. By definition, $\nu^{(g,h)}_n(t)$ and $\kappa^{\varphi,(g,h)}(t)$ are real-analytic, and since the convergence in Equation (10.7) is uniform, we can intertwine limit and derivative to obtain the desired result. □

10.2. **Proof of Theorem 10.2.** We place ourselves under the assumptions of Theorem 10.2 and begin with the following lemma that does not assume Zariski-density of the base point, it will be also needed later on.

Lemma 10.9. *Consider an analytic curve $(\rho_t:\Gamma\to\mathsf{G})_{t\in(-\varepsilon,\varepsilon)}$ with speed v and a loxodromic $g\in\Gamma$. Consider $\varphi\in(\mathfrak{a}_\vartheta)^*$ and assume there exists $\alpha\in\operatorname{supp}\varphi$ with $\dim\mathfrak{g}_\alpha=1$ that strictly minimizes g among $\operatorname{supp}\varphi$. Let $h\in\Gamma$ be such that the pair (g,h) is strongly transversally ϑ-proximal. Consider $\psi\in(\mathfrak{a}_\vartheta)^*$ and assume that the values $\sigma^g(\rho)$, for $\sigma\in\operatorname{supp}\psi$, are all distinct. If for all loxodromic $\gamma\in\Gamma$ it holds $\mathrm{d}\varphi^\gamma(v)-\psi^\gamma(\rho)\in\mathbb{Z}$, then*

$$\mathrm{d}\alpha^g(v)=0.$$

Proof. By definition (Eq. (10.5)), the vector $\nu_n(g,h)$ is a uniform double limit of sums of vectors of the form $\pm\lambda(\rho(g^n(g^kh^l)^m))$. Moreover, for every n, $\nu_n(g,h)$ is an analytic function on g and h, which is also a uniform limit of analytic functions. Since the curve ρ_t is analytic we can intertwine limit and derivative in the definition of ν_n so for every $n\in\mathbb{N}$ it holds, as g and h are transversally ϑ-proximal, that

$$m_n:=\mathrm{d}\varphi\big(\nu^{\vartheta,(g,h)}_n\big)(v)-\psi\big(\nu^{\vartheta,(g,h)}_n(\rho)\big)\in\mathbb{Z}, \tag{10.10}$$

where we have denoted by $\nu_n^{\vartheta,(g,h)}(\rho_t) = \nu_n^{\vartheta}(\rho_t g, \rho_t h)$. Since $\dim \mathfrak{g}_\alpha = 1$ Corollary 10.8 states that

$$\mathrm{d}\varphi\big(\nu_{2n}^{\vartheta,(g,h)}\big)(v)e^{2n\alpha^g(\rho)} + \varphi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big)2n\big(\mathrm{d}\alpha^g(v)\big)e^{2n\alpha^g(\rho)} \xrightarrow[n\to\infty]{} \mathrm{d}\kappa^{\varphi,(g,h)}(v).$$

Pairing with Equation (10.10) gives

$$\Big(\psi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big)+m_{2n}\Big)e^{2n\alpha^g(\rho)}+\varphi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big)2n\big(\mathrm{d}\alpha^g(v)\big)e^{2n\alpha^g(\rho)} \xrightarrow[n\to\infty]{} \mathrm{d}\kappa^{\varphi,(g,h)}(v). \tag{10.11}$$

Dividing by n and considering the limit we obtain by Equation (10.7)

$$\Big(\psi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big) + m_{2n}\Big)\frac{e^{2n\alpha^g(\rho)}}{n} \xrightarrow[n\to\infty]{} 2\mathrm{d}\alpha^g(v)\kappa^{\varphi,(g,h)}(\rho) \tag{10.12}$$

One has that $\nu_{2n}^{\vartheta,(g,h)}(\rho) \to 0$ and thus also does $\psi(\nu_{2n}^{\vartheta,(g,h)}(\rho))$. Since $e^{2n\alpha^g(\rho)}/n$ is divergent, we obtain that $m_{2n} = 0$ for all big enough n, giving in turn that

$$\psi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big)\frac{e^{2n\alpha^g(\rho)}}{n} \xrightarrow[n\to\infty]{} 2\mathrm{d}\alpha^g(v)\kappa^{\varphi,(g,h)}(\rho). \tag{10.13}$$

Using the definition of $\nu_n^{\vartheta,(g,h)}$ we obtain

$$e^{2n\alpha^g(\rho)}\psi\big(\nu_{2n}^{\vartheta}(g,h)\big) = \sum_{\sigma\in\vartheta} \psi_\sigma e^{2n\left(\alpha^g(\rho)-\sigma^g(\rho)\right)} e^{2n\sigma^g(\rho)} \log\big|c_{2n}(\phi_\sigma g, \phi_\sigma h)\big|.$$

If there exists $\sigma \in \operatorname{supp}\psi$ so that $\alpha^g(\rho)-\sigma^g(\rho) > 0$ then we let σ be the root that maximizes this value. Applying Lemma 10.6 to the representation ϕ_σ we obtain a subsequence n_k such that

$$e^{2n_k\sigma^g(\rho)} \log\big|c_{2n_k}(\phi_\sigma g, \phi_\sigma h)\big| \xrightarrow{k\to\infty} K \neq 0.$$

Since the terms $e^{2n_k\delta^g(\rho)} \log\big|c_{2n_k}(\phi_\delta g, \phi_\delta h)\big|$ are bounded for every $\delta \in \operatorname{supp}\psi$, we deduce that $e^{2n_k\alpha^g(\rho)}\psi\big(\nu_{2n_k}^{\vartheta}(g,h)\big)$ is diverging to infinity at an exponential rate $\mu = \alpha^g(\rho) - \sigma^g(\rho) > 0$, which combined with Equation (10.13) gives $e^{2n_k\mu}/n_k$ is convergent as $k \to \infty$, a contradiction.

We conclude that $\forall\ \sigma \in \operatorname{supp}\psi$ one has $\alpha^g(\rho) \leq \sigma^g(\rho)$ and applying Lemma 10.6 we obtain, since $e^{2n\sigma^g(\rho)} \log\big|c_{2n}(\phi_\sigma g, \phi_\sigma h)\big|$ is bounded for every σ, that $e^{2n\alpha^g(\rho)}\psi\big(\nu_{2n}^{\vartheta}(g,h)\big)$ is converging to a constant C, which is possibly zero.

Thus

$$\psi\big(\nu_{2n}^{\vartheta,(g,h)}(\rho)\big)\frac{e^{2n\alpha^g(\rho)}}{n} \xrightarrow[n\to\infty]{} 0,$$

giving, since $\kappa^\varphi(g,h) \neq 0$ by Proposition 10.7, that $\mathrm{d}\alpha^g(v) = 0$ as desired. ☐

Proof of Theorem 10.2. Let us assume by contradiction that for all $\gamma \in \Gamma$ one has

$$\mathrm{d}\varphi^\gamma(v) - \psi^\gamma(\rho) \in \mathbb{Z}.$$

By hypothesis, there exists $\alpha \in \operatorname{supp}\varphi$ with $\dim \mathfrak{g}_\alpha = 1$ and a loxodromic γ verifying Eq. (10.1). The Zariski-density assumption gives (c.f. Benoist-Quint [5, Lemma 7.20]), for each loxodromic γ, an $h \in \Gamma$ such that (γ, h) are strongly transversally loxodromic. Thus, we can apply Lemma 10.9 to every loxodromic $g \in \Gamma$ such that the values $\sigma^g(\rho)$, for $\sigma \in \vartheta$, are all distinct and verifies Equation (10.1) to obtain:

Remark 10.10. For every $g \in \Gamma$ loxodromic such that the values $\sigma^g(\rho)$, for $\sigma \in \vartheta$, are all distinct and verifying Eq. (10.1) one has $\mathrm{d}\alpha^g(v) = 0$.

Proposition 9.1 states we can choose $\gamma\in\Gamma_{\text{full}}$ verifying the assumptions of the above remark. Moreover, we use Proposition 2.15 by Benoist and more specifically Remark 2.16 to choose a Zariski-dense sub-semi-group $\Gamma'<\Gamma$ that contains γ^k for some large power k and whose limit cone is a convex cone about $\mathbb{R}_+\lambda(\rho\gamma)$, chosen so that for all $h\in\Gamma'$ Equation (10.1) holds (for h instead of γ) and such that the values $\sigma^h(\rho)$, for $\sigma\in\vartheta$, are all distinct.

Since Γ' is chosen with $\gamma^k\in\Gamma'$, the curve $\eta_t:=\rho_t|\Gamma'$ has full loxodromic variation. However, Remark 10.10 gives that

$$\forall h\in\Gamma' \text{ it holds } \mathrm{d}\alpha^h(\vec{\eta})=0,$$

contradicting Theorem A. This completes the proof. □

11. The case of ϑ-Anosov representations: cohomological independence and other consequences

We fix throughout this section a word-hyperbolic group Γ and an analytic curve $(\rho_t)_{t\in(-\varepsilon,\varepsilon)}\subset\mathfrak{A}_\vartheta(\Gamma,\mathsf{G})$ with $\rho_0=\rho$ and $v=\vec{\rho}$.

11.1. **Full loxodromic variation for ϑ-Anosov representations.** In this paragraph we establish the following.

Corollary 11.1. *Assume that $\vartheta\cap\Delta_i\neq\emptyset$ for every simple factor $\mathfrak{g}_i$ of $\mathfrak{g}$. If v has full variation then it has (full) loxodromic variation.*

The proof is essentially contained in Bridgeman-Canary-Labourie-S. [14, § 10] so we only give the minor required modifications.

Lemma 11.2. *Let $\rho:\Gamma\to\mathsf{G}$ be ϑ-Anosov and have Zariski-dense image, then the set of fixed points*

$$\{(h^-,h^+,\gamma^-,\gamma^+):\rho(\gamma)\text{ and }\rho(h)\text{ are loxodromic}\}$$

is dense in $\partial^{(4)}\Gamma$.

Proof. The Lemma is certainly true if we remove the 'loxodromic' condition, we show how we reduce the question to this situation. Since $\rho(\Gamma)$ is Zariski-dense $\xi^\vartheta(\partial\Gamma)$ is the limit set of $\rho(\Gamma)$ on the flag space $\mathcal{F}_\vartheta(\mathsf{G})$. By Benoist [3, Remarque 3.6 2)] it is the image, under the natural projection, of the limit set $\Lambda_{\rho(\Gamma)}$ in the full flag space $\mathcal{F}_\Delta(\mathsf{G})$. Again by Zariski-density, the latter is the closure of attracting full flags of loxodromic elements in $\rho(\Gamma)$, so the Lemma follows. A standard ping-pong argument gives then the result. □

Proof of Corollary 11.1. Consider $\alpha_i\in\vartheta\cap\Delta_i$. We will show that for each i there exist $\gamma\in\Gamma$ with loxodromic $\rho(\gamma)$ such that one has $\big(\partial/\partial t\big)|_{t=0}\varpi_{\alpha_i}(\lambda(\rho_t(\gamma)))\neq0$. Otherwise, for every $\gamma\in\Gamma$ with loxodromic $\rho(\gamma)$ one has

$$\frac{\partial}{\partial t}\Big|_{t=0}\lambda_1\big(\phi_{\alpha_i}(\rho_t(\gamma))\big)=0.$$

Using now [14, Prop. 9.4] one has that for every co-prime pair $\gamma,h\in\Gamma$ with $\rho(\gamma)$, $\rho(h)$ loxodromic, it holds $\partial^{\log}\mathrm{b}_{\phi_{\alpha_i}\rho_t}(\gamma^-,\gamma^+,h^-,h^+)=0$ (recall notation from §2.13.5).

By Lemma 11.2, this implies that for every four-tuple of points $(x,y,z,t)\in\partial^{(4)}\Gamma$ one has $\partial^{\log}\mathrm{b}_{\phi_{\alpha_i}\rho_t}(\gamma^-,\gamma^+,h^-,h^+)=0$.

Moreover $\phi_{\alpha_i}\rho$ is irreducible and projective Anosov, so from this point on the proof of [14, Lemma 10.3] woks verbatim to give that the cocycle $p_i(\mathsf{u}_v)$ is cohomologically trivial, contradicting our assumptions. □

11.2. **Cohomological independence of $\mathcal{J}^\vartheta$ and $\vec{\mathcal{J}}^\vartheta$.** Consider the Ledrappier potential $\mathcal{J}_t = \mathcal{J}^\vartheta_{\rho_t} : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$ and denote by

$$\vec{\mathcal{J}} = \vec{\mathcal{J}}^\vartheta : \mathsf{U}\Gamma \to \mathfrak{a}_\vartheta$$

its differentiation w.r.t. t at 0. Theorem 10.2 together with Livšic's Theorem readily imply the following and Corollary 11.1.

Corollary 11.3. *Let $v \in \mathsf{T}_\rho\mathfrak{A}_\vartheta(\Gamma,\mathsf{G})$ have full variation and Zariski-dense basepoint.*

(i) *Consider $\varphi,\psi \in (\mathfrak{a}_\vartheta)^*$ and assume there exist $\gamma \in \Gamma$ and a multiplicity-1 root α that strictly minimizes γ among $\operatorname{supp}\varphi$. Then, $\varphi(\vec{\mathcal{J}}^\vartheta) - \psi(\mathcal{J}^\vartheta)$ is not Livšic-cohomologous to a function with periods in $\mathbb{Z}$.*

(ii) *If moreover every root in ϑ has multiplicity one then $\mathcal{J}^\vartheta$ and $\vec{\mathcal{J}}^\vartheta$ are Livšic-cohomologically independent (thus Corollary 2.35 applies).*

Recall from Remark 2.33 that if $\mathbf{P}^\psi$ is degenerate at v, then the set of normalized variations $\mathbb{V}^\psi_{\vartheta,v}$ is contained in a level set of ψ. This is a much weaker condition than having non-empty interior and thus we obtain non-degeneracy of $\mathbf{P}^\psi_\rho$ in more situations.

Corollary 11.4. *Consider $\psi \in \operatorname{int}(\mathcal{L}_{\vartheta,\rho})^*$ and assume there exist $\gamma \in \Gamma$ and α with $\dim\mathfrak{g}_\alpha = 1$ that strictly minimizes γ among $\operatorname{supp}\psi$. If $v \in \mathsf{T}_\rho\mathfrak{X}(\Gamma,\mathsf{G})$ has full variation then $\mathbf{P}^\psi_\rho(v) > 0$.*

Proof. By Theorem 2.19, $\mathbf{P}^\psi_\rho$ degenerates at v iff $(\mathrm{d}\hbar^\psi v)\psi(\mathcal{J})$ and $\hbar^\psi\psi(\vec{\mathcal{J}})$ are Livšic-cohomologous. However this does not hold by Corollary 11.3(i). □

11.3. **Variations along level sets of $\hbar$ give non-proper actions.** The principle indicated in the title is used in Labourie [47, 51], we give new situations where it applies.

Let us recall from Sullivan [73] (see also Yue [77]), that if X has rank 1 and $\rho : \Gamma \to \operatorname{Isom}\mathsf{X}$ is convex-co-compact, then $\hbar_\rho$ is the Hausdorff dimension of the limit set of ρ on the visual boundary of X for a visual metric. Recall also from Bridgeman-Canary-Labourie-S. [14] (see § 2.13.2) that the function $\rho \mapsto \hbar_\rho$ is real-analytic on the of convex-co-compact representations on X.

The adjoint representation of a rank-one simple $\mathfrak{g}$ has neutralizing dimension 1 as long as $\mathfrak{Z}(\mathfrak{m}) = \{0\}$. Thus, the rank 1 simple groups with $\operatorname{neudim}(\mathrm{Ad}) \neq 1$ have Lie algebras equal to $\mathfrak{so}_{1,3}$ or $\mathfrak{su}_{1,n}$ for $n \geq 2$ (see Knapp [44, Appendix C]), whence one concludes the following:

Corollary 11.5 (Rank 1)**.** *Let G be the identity component of the isometry group of $\mathbb{H}^n_{\mathbb{R}}$ for $n \neq 3$, $\mathbb{H}^n_{\mathbb{H}}$ $n \geq 2$, or the Cayley hyperbolic plane, and let $\rho : \Gamma \to \mathsf{G}$ be convex co-compact and Zariski-dense. Let $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\Gamma,\mathfrak{g})$ be an integrable cocycle. If $\mathrm{d}_\rho\hbar(\mathsf{u}) = 0$ then the affine action of $\rho(\Gamma)_{\mathsf{u}}$ on $\mathfrak{g}$ is not proper. In particular:*

$$\hbar \text{ is critical at } \rho \Rightarrow \text{there is no proper action on } \mathfrak{g} \text{ above } \mathrm{Ad}\,\rho.$$

Proof. If $\mathrm{d}\hbar(\mathsf{u}) = 0$ Eq. (2.19) implies that $\varpi_\alpha(p_{\varpi_\alpha}\mathsf{u}) = 0$. Lemma 2.36 gives then $0 \in \operatorname{int}\mathbb{V}_{\mathsf{u}}^{\phi^\psi}$. Since $\mathfrak{Z}(\mathfrak{m}) = \{0\}$ Proposition 8.1 implies that $\mathbb{V}_{\mathsf{u}}^{\phi^\psi} = \mathrm{MS}^{\phi^\psi}(\mathsf{u})$. Kassel-Smilga's Proposition 7.2 gives then non-properness of the corresponding action. □

Corollary 11.6. *We let G be as in Corollary 11.5 and $\mathbb{F}$ be a non-abelian free group, then there exists $C > 0$ such that if $\rho : \mathbb{F} \hookrightarrow \mathsf{G}$ is a Schottky group with contraction greater than C, then $\hbar$ is not critical at ρ.*

Proof. Follows from Corollary 11.5 together with Smilga's construction [70]. □

Let us consider now $\mathsf{G} = \mathsf{SL}(3,\mathbb{R})$ and the functional $\mathsf{H} \in \mathfrak{a}^*$ given by $\mathsf{H}(a) = (a_1 - a_3)/2$, whose associated entropy $\hbar^{\mathsf{H}} : \mathfrak{A}_\Delta(\Gamma, \mathsf{SL}(3,\mathbb{R})) \to \mathbb{R}_+$ is usually called the *Hilbert entropy*.

Corollary 11.7 ($\mathsf{SL}(3,\mathbb{R})$)**.** *Consider a non-zero $v \in \mathsf{T}_\rho\mathfrak{A}_\Delta(\Gamma, \mathsf{SL}(3,\mathbb{R}))$ with Zariski-dense base-point. If $\mathrm{d}\hbar^{\mathsf{H}}(v) = 0$ then the affine action on $\mathfrak{sl}(3,\mathbb{R})$ via u_v is not proper. In particular if $\hbar^{\mathsf{H}}$ is critical at ρ, then there exists a neighborhood $\mathcal{U}$ of ρ such that there is no proper affine action of Γ on $\mathfrak{sl}(3,\mathbb{R})$ above any $\operatorname{Ad}\eta$ for $\eta \in \mathcal{U}$.*

Proof. Since $\mathsf{SL}(3,\mathbb{R})$ is simple, $v \neq 0$ is equivalent to having full variation. Applying Corollary 11.3 we obtain that $\mathbb{V}_v^{\mathsf{H}}$ has non-empty interior and by Corollary 2.35

$$p_{\mathsf{H}}v \in \operatorname{int}\mathbb{V}_v^{\mathsf{H}}.$$

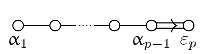
By Equation (2.19) $\mathsf{H}(p_{\mathsf{H}}v) = -\mathrm{d}\log\hbar^{\mathsf{H}}(v) = 0$, so $p_{\mathsf{H}}v \in \ker\mathsf{H}$. Since H is i-invariant, the set of normalized variations $\mathbb{V}_v^{\mathsf{H}}$ is also i-invariant so we obtain $\mathrm{i}(p_{\mathsf{H}}v) = -p_{\mathsf{H}}v \in \operatorname{int}\mathbb{V}_v^{\mathsf{H}}$, thus by convexity, $0 \in \operatorname{int}\mathbb{V}_v^{\mathsf{H}}$. Since $\mathsf{SL}(3,\mathbb{R})$ is split, $\mathfrak{m} = 0$ and in particular $\mathfrak{Z}(\mathfrak{m}) = \{0\}$, so by Proposition 8.1 we conclude that $0 \in \operatorname{int}\mathrm{MS}^{\phi^\psi}(\mathsf{u}_v)$. Kassel-Smilga's Proposition 7.2 gives then non-properness of the corresponding action. By continuity of $v \mapsto \mathbb{V}_v^{\mathsf{H}}$ the same conclusion holds on a neighborhood of v in $\mathsf{T}\mathfrak{X}(\Gamma, \mathsf{SL}(3,\mathbb{R}))$, completing the proof. □

As before, Corollary 11.7 together with Smilga's construction [70] gives:

Corollary 11.8. *Let $\mathbb{F}$ be a non-abelian free group, then there exists $C > 0$ such that if $\rho : \mathbb{F} \hookrightarrow \mathsf{SL}(3,\mathbb{R})$ is a Schottky subgroup with contraction greater than C, then $\hbar^{\mathsf{H}}$ is not critical at ρ.*

12. The case of Θ-positive representations

The restricted root system of the group $\mathsf{SO}(p,q)$, for $p < q$, has Dynkin diagram



.

All the long roots on this diagram have one-dimensional root spaces (see the Appendix in Knapp [44]). We let $\Theta = \{\alpha_1, \dots, \alpha_{p-1}\}$. Guichard-Wienhard [36] have introduced the notion of a Θ-*positive representation* from $\pi_1 S$ with values in $\mathsf{SO}(p,q)$. We refer to their work for the definition. Instead, we will use the following result, which states that these representations are Θ-Anosov, and verify an even stronger form called *hyperconvexity*. We will whence study in §12.1 hyperconvex representations and then come back to Θ-positive representations.

Let us denote by $\mathcal{P}_\Theta^{p,q}(S)$ the space of Θ-positive representations of S with values in $\mathsf{SO}(p,q)$. Item (i) below places this setting in that earlier sections of this paper.

Theorem 12.1.

(i) (Beyrer-Pozzetti[7], Guichard-Labourie-Wienhard [34]) *Every* Θ*-positive representation is* Θ*-Anosov, moreover,* $\mathcal{P}^{p,q}_{\Theta}(S) \subset \mathfrak{X}\big(\pi_1 S, \mathsf{SO}(p,q)\big)$ *is open and closed.*

(ii) (Pozzetti-S.-Wiehard [63, Theorem 10.3]) *For every* $k \in [\![1, p-2]\!]$ *the representation* $\Lambda^k \rho$ *is* $(1,1,2)$*-hyperconvex (see Definition 12.2 below). In particular,* $\hbar^{\alpha_k}_{\rho} = 1$.

12.1. $(1,1,2)$**-hyperconvex representations and Hausdorff dimension.** The main purpose of this section is to establish Corollary 12.6 below.

We recall a definition from Pozzetti-S.-Wienhard [62]. We consider $\mathsf{SL}(d,\mathbb{C})$ as a real-algebraic Lie group.

Definition 12.2. A representation $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{C})$ is $(1,1,2)$-hyperconvex if it is $\{\sigma_1, \sigma_2\}$-Anosov and for every triple $x, y, z \in \partial\Gamma$ of pairwise distinct points one has

$$\big(\xi^1(x) \oplus \xi^2(y)\big) \cap \xi^{d-2}(z) = \{0\}.$$

Let us denote by

$$\mathfrak{A}^{\pitchfork}_{\{\sigma_1,\sigma_2\}}\big(\Gamma, \mathsf{SL}(d,\mathbb{C})\big) = \big\{\rho : \Gamma \to \mathsf{SL}(d,\mathbb{C}) \text{ is } (1,1,2)\text{-hyperconvex}\big\},$$

it is an open subset of the character variety $\mathfrak{X}\big(\Gamma, \mathsf{SL}(d,\mathbb{C})\big)$ (Pozzetti-S.-Wienhard [62, Proposition 6.2]).

Theorem 12.3. *Let* $\partial\Gamma$ *be homeomorphic to a circle and* $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ *a* $(1,1,2)$*-hyperconvex representation, then*

(i) (Pozzetti-S.-Wienhard [62]) $\hbar^{\sigma_1}_{\rho} = 1$ *and*

(ii) (Pozzetti-S. [61, Theorem C]) *if* $\rho(\Gamma)$ *acts furthermore irreducibly on* $\mathbb{R}^d$ *then the Zariski closure* G *of* $\rho(\Gamma)$ *is simple and the highest restricted weight of the representation* $\mathsf{G} \hookrightarrow \mathsf{SL}(d,\mathbb{R})$ *is a multiple of a fundamental weight for a root* $\alpha \in \Delta$ *with* $\dim \mathfrak{g}_{\alpha} = 1$.

Corollary 12.4. *Let* $\partial\Gamma$ *be homeomorphic to a circle and* $\rho : \Gamma \to \mathsf{SL}(d,\mathbb{R})$ *be an irreducible* $(1,1,2)$*-hyperconvex representation with Zariski closure* G*. Let* $\alpha \in \Delta_{\mathsf{G}}$ *be the root such that the representation* $\mathsf{G} \hookrightarrow \mathsf{SL}(d,\mathbb{R})$ *has highest weight* $n\varpi_{\alpha}$*. Then* $\mathbf{P}^{\alpha}$ *is (well defined and) Riemannian on* $\mathsf{T}_{\rho}\mathfrak{X}(\Gamma, \mathsf{G})$*. Moreover, considering* $\mathfrak{X}(\Gamma, \mathsf{G}) \subset \mathfrak{X}(\Gamma, \mathsf{SL}(d,\mathbb{R}))$ *one has* $\mathbf{P}^{\sigma_1}(v) = \mathbf{P}^{\alpha}(v)$ *for every* $v \in \mathsf{T}_{\rho}\mathfrak{X}(\Gamma, \mathsf{G})$*.*

Proof. Denote by $\phi : \mathsf{G} \to \mathsf{SL}(d,\mathbb{R})$ the representation induced by the inclusion. By Theorem 12.3 it has highest restricted weight $n\varpi_{\alpha}$ for some $n \in \mathbb{N}$. Whence $\forall\, g \in \mathsf{G}$

$$\begin{aligned}
\sigma_1\big(\lambda(\phi g)\big) &= \alpha\big(\lambda_{\mathsf{G}}(g)\big),\\
\sigma_2\big(\lambda(\phi g)\big) &= \min_{\sigma \in \Delta_{\mathsf{G}} : \langle \alpha, \sigma\rangle \neq 0} \sigma\big(\lambda_{\mathsf{G}}(g)\big).
\end{aligned}$$

Since ρ is $\{\sigma_2\}$-Anosov it follows from this last equation that if we let $\vartheta = \{\sigma \in \Delta_{\mathsf{G}} : \langle \alpha, \sigma \rangle \neq 0\}$ be the set of roots neighboring α in the Dynkin diagram of G, then $\rho : \Gamma \to \mathsf{G}$ is ϑ-Anosov. Moreover $\alpha \in \operatorname{int} \vartheta$ and in particular $\alpha \in (\mathfrak{a}_{\vartheta})^*$. It follows that the pressure form $\mathbf{P}^{\alpha}$ is well defined and one has $\mathbf{P}^{\sigma_1}_{\rho}(v) = \mathbf{P}^{\alpha}_{\rho}(v)$. Since $\mathfrak{A}^{\pitchfork}_{\{\sigma_1,\sigma_2\}}\big(\Gamma, \mathsf{SL}(d,\mathbb{R})\big)$ is open and, by Theorem 12.3, $\rho \mapsto \hbar^{\sigma_1}_{\rho}$ is constant on this space, it follows that $\hbar^{\alpha}$ is critical at ρ and thus $\mathbf{P}^{\alpha}$ is degenerate at $v \in \mathsf{T}_{\rho}\mathfrak{X}(\Gamma, \mathsf{G})$ if and only if for every $\gamma \in \Gamma$ one has $\alpha(\mathrm{d}\lambda^{\gamma}(v)) = 0$ (Eq. (2.11)). However, since

G is simple and $v \neq 0$, it has full variation and, since ρ is Anosov Corollary 11.1 implies that v has full loxodromic variation. We can thus apply Corollary 8.4 to obtain that there exists $\gamma \in \Gamma$ such that $\alpha(\mathrm{d}\lambda^\gamma(v)) \neq 0$, thus $\mathbf{P}^\alpha(v) > 0$. □

12.1.1. *Root system of the complexification.* We recall here some elementary facts needed in subsection 12.1.2. Let $\mathfrak{g}$ be a simple real Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.

Let $\mathfrak{g}_\mathbb{C}$ be the complexification of $\mathfrak{g}$, then $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ is a compact subalgebra and $\mathfrak{s} = \mathfrak{p} \oplus i\mathfrak{k}$ is an $\mathrm{ad}_\mathfrak{u}$-module. It follows that the involution $\tau : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ defined as $\tau|\mathfrak{u} = \mathrm{id}$ and $\tau|\mathfrak{s} = -\mathrm{id}$ is a Lie algebra involution which is moreover a Cartan involution of $\mathfrak{g}_\mathbb{C}$.

Let $\mathfrak{b} \subset \mathfrak{m}$ be a maximal Abelian subalgebra. Then $\mathfrak{h} = \mathfrak{a} \oplus \mathfrak{b}$ is a Cartan subalgebra of $\mathfrak{g}$, meaning that $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$. As such, we have a root-space decomposition

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha \in \Sigma} (\mathfrak{g}_\mathbb{C})_\alpha,$$

where

$$(\mathfrak{g}_\mathbb{C})_\alpha = \big\{x \in \mathfrak{g}_\mathbb{C} : \forall h \in \mathfrak{h}_\mathbb{C} \text{ one has } [h, x] = \alpha(h)x\big\},$$
$$\Sigma = \big\{\alpha \in (\mathfrak{h}_\mathbb{C})^* : (\mathfrak{g}_\mathbb{C})_\alpha \neq \{0\}\big\}.$$

Corollary 6.49 from Knapp's book [44] states that every $\alpha \in \Sigma$ verifies $\alpha|\mathfrak{a} \oplus i\mathfrak{b}$ is real-valued, so since $\mathfrak{a} \oplus i\mathfrak{b}$ is a real form form of $\mathfrak{h}_\mathbb{C}$, α is uniquely determined by $\alpha|\mathfrak{a} \oplus i\mathfrak{b}$. Moreover $\mathfrak{a} \oplus i\mathfrak{b}$ is a maximal abelian subspace of $\mathfrak{s}$, so $\Sigma' = \{\alpha|\mathfrak{a} \oplus i\mathfrak{b} : \alpha \in \Sigma\}$ is the restricted root system of $\mathfrak{g}_\mathbb{C}$ as a real Lie algebra of non-compact type.

One has also that ([44, Eq. 6.48b]) if $\sigma \in \Phi$ then

$$\mathfrak{g}_\sigma = \mathfrak{g} \cap \Big(\bigoplus_{\alpha \in \Sigma : \alpha|\mathfrak{a} = \sigma} (\mathfrak{g}_\mathbb{C})_\alpha \Big).$$

We obtain thus the following Lemma.

Lemma 12.5. *Let $\mathfrak{g}$ be simple and assume there exists $\sigma \in \Phi$ such that $\dim \mathfrak{g}_\sigma = 1$, then $\mathfrak{g}_\mathbb{C}$ is simple and there exists a unique $\alpha_\sigma \in \Sigma$ such that $\alpha_\sigma|\mathfrak{a} = \sigma$. Consequently, if $\sigma \in \Delta$, we have a natural embedding of flag spaces $\mathcal{F}_{\{\sigma\}}(\mathsf{G}) \subset \mathcal{F}_{\{\alpha_\sigma\}}(\mathsf{G}_\mathbb{C})$.*

Proof. The real algebra $\mathfrak{g}$ cannot in itself be complex (otherwise every root space has dimension $\dim_\mathbb{R} = 2$), so the first statement follows from [44, Theorem 9.4(b)].

Concerning the second statement, for every $\alpha \in \Sigma$ one has $(\mathfrak{g}_\mathbb{C})_\alpha \cap \mathfrak{g} \neq 0$, indeed if $x + iy \in (\mathfrak{g}_\mathbb{C})_\alpha$ with $x, y \in \mathfrak{g}$, then

$$[a, x + iy] = [a, x] + i[a, y] = \alpha(a)x + \alpha(a)iy.$$

So since $\alpha|\mathfrak{a}$ is real valued one has, for all $a \in \mathfrak{a}$, $[a, x] = \alpha(a)x$ and $[a, y] = \alpha(a)y$. The last assertion follows. □

12.1.2. *Variation of Hausdorff dimension on complex groups.* Throughout this subsection we will work with the real and complex forms of a given simple real-algebraic Lie group, to avoid confusion we will let $\mathsf{G}_\mathbb{R}$ be the real points and $\mathsf{G}_\mathbb{C}$ be the group of its $\mathbb{C}$-points. Consider the natural embedding $\mathsf{G}_\mathbb{R} \subset \mathsf{G}_\mathbb{C}$.

If α is a restricted root of $\mathsf{G}_\mathbb{C}$ (as a real group), then we fix a Riemannian metric on the flag space $\mathcal{F}_{\{\alpha\}}(\mathsf{G}_\mathbb{C})$, denote by $\mathrm{Hff}(X)$ the associated Hausdorff dimension of a subset X of $\mathcal{F}_{\{\alpha\}}(\mathsf{G}_\mathbb{C})$ and consider the function

$$\begin{aligned}\mathrm{Hff}_\alpha : \mathfrak{A}_{\{\alpha\}}(\Gamma, \mathsf{G}_\mathbb{C}) &\to \mathbb{R}_{>0}\\ \rho &\mapsto \mathrm{Hff}\big(\xi^\alpha(\partial\Gamma)\big).\end{aligned}$$

We emphasize that the action of $\mathsf{G}_\mathbb{C}$ on $\mathcal{F}_{\{\alpha\}}(\mathsf{G}_\mathbb{C})$ is not conformal for the chosen metric (unless $\mathsf{G}_\mathbb{C} = \mathsf{SL}(2,\mathbb{C})$)

By Lemma 12.5, if $\sigma \in \Delta$ is a restricted root of $\mathsf{G}_\mathbb{R}$ so that $\dim \mathfrak{g}_\sigma = 1$, then there is a unique restricted root α_σ of $\mathsf{G}_\mathbb{C}$ so that $\alpha_\sigma|\mathfrak{a} = \sigma$, so we have an inclusion

$$\mathfrak{A}_{\{\sigma\}}(\Gamma, \mathsf{G}_\mathbb{R}) \subset \mathfrak{A}_{\{\alpha_\sigma\}}(\Gamma, \mathsf{G}_\mathbb{C}),$$

and an embedding of the flag spaces $\mathcal{F}_{\{\sigma\}}(\mathsf{G}_\mathbb{R}) \subset \mathcal{F}_{\{\alpha_\sigma\}}(\mathsf{G}_\mathbb{C})$.

In this section we establish the following.

Corollary 12.6. *Let $\mathsf{G}_\mathbb{R}$ be simple, real-algebraic and connected and consider $\alpha \in \Delta$ with $\dim \mathfrak{g}_\alpha = 1$. Let $\rho : \pi_1 S \to \mathsf{G}_\mathbb{R}$ have Zariski-dense image and be such that $\phi_\alpha\rho$ is $(1,1,2)$-hyperconvex. Then for every $v \in \mathsf{T}_\rho\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C})$ that is not tangent to the real characters one has*

$$\mathrm{Hess}_\rho \mathrm{Hff}_\alpha(v) > 0.$$

In particular, there exists a neighborhood $\mathcal{V}$ of ρ inside the complex characters $\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C})$ such that if $\eta \in \mathcal{V}$ verifies

$$\mathrm{Hff}\big(\xi^\alpha_\eta(\partial\pi_1 S)\big) = 1$$

then the Zariski closure of $\eta(\pi_1 S)$ is (conjugate to) $\mathsf{G}_\mathbb{R}$.

The remainder of the section is devoted the proof of the Corollary.

Proof. Let J denote the almost-complex structure of the complex characters

$$\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C})$$

induced by the complex structure of $\mathsf{G}_\mathbb{C}$. Let us also consider the irreducible representation $\phi_\alpha : \mathsf{G}_\mathbb{R} \to \mathsf{SL}(d,\mathbb{R})$ which extends by complexifying to a representation $\phi_\alpha : \mathsf{G}_\mathbb{C} \to \mathsf{SL}(d,\mathbb{C})$. Since $\phi_\alpha\mathsf{G}_\mathbb{R}$ contains a proximal element, the complexified representation is also irreducible (over $\mathbb{C}$). Theorem 2.27 implies then that ρ is a regular point of $\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C})$ and thus, since ρ has values in $\mathsf{G}_\mathbb{R}$, the tangent space splits

$$\mathsf{T}_\rho\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C}) = \mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R}) \oplus \mathsf{J}\big(\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})\big). \tag{12.1}$$

Moreover since representation $\phi_\alpha\circ\rho$ is $(1,1,2)$-hyperconvex, Bridgeman-Pozzetti-S.-Wienhard [16, Theorem A] applies to give that states that if $v \in \mathsf{T}_\rho\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$ is non-zero, then

$$\mathrm{Hess}_\rho \mathrm{Hff}_{\sigma_1}(\mathsf{J}v) = \mathbf{P}^{\sigma_1}_\rho(v).$$

Thus combining with Corollary 12.4 we obtain that, for every $v \in \mathsf{T}_\rho\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{R})$

$$\mathrm{Hess}_\rho \mathrm{Hff}_\alpha(\mathsf{J}v) > 0.$$

It follows that any $w \in \mathsf{T}_\rho\mathfrak{X}(\pi_1 S, \mathsf{G}_\mathbb{C})$ that is not tangent to the real characters, verifies $\mathrm{Hess}_\rho \mathrm{Hff}_\alpha(w) > 0$ and thus the result follows. □

12.2. **Hessian of Hausdorff dimension.** We now focus on Θ-positive representations. Theorem 12.1(ii) by Pozzetti-S.-Wienhard, together with Corollary 12.6 readily give:

Corollary 12.7. *For every non-zero $v \in \mathsf{T}\mathcal{P}^{p,q}_\Theta(S)$ with Zariski-dense basepoint, and $\upalpha \in \mathrm{int}\,\Theta$ one has $\mathrm{Hess}_\rho \mathrm{Hff}_\upalpha(\mathsf{J}v) > 0$.*

12.3. **Length functions and pressure.** By means of Theorem 12.1 and Corollary 11.4 we establish the following:

Corollary 12.8. *Let $\rho : \pi_1 S \to \mathsf{SO}(p,q)$ be Θ-positive and denote by G the Zariski closure of $\rho(\pi_1 S)$.*

(i) *If $\mathsf{G} = \mathsf{SO}(p,q)$ then $\forall\, \psi \in \operatorname{int}(\mathcal{L}_{\Theta,\rho})^*$ the pressure form $\mathbf{P}^{\psi}_{\rho}$ is Riemannian.*
(ii) *If $\rho(\pi_1 S)$ acts irreducibly on $\mathbb{R}^{p+q}$, then for every $\psi \in \operatorname{int}(\mathcal{L}_{\operatorname{int}\Theta,\rho})^*$ one has $\mathbf{P}^{\psi}$ is Riemannian when restricted to characters with values in G.*

Proof. When $\rho(\pi_1 S)$ is Zariski-dense then, since all roots in Θ have one-dimensional root spaces and $\mathsf{SO}(p,q)$ is a simple group, the result readily follows from Corollary 11.4. The second item is a bit more involved. Since $\rho(\pi_1 S)$ acts irreducibly, the combination of Theorem 12.1(ii) and Theorem 12.3 by Pozzetti-S. gives that G is simple. Moreover, the combination of Theorem 12.1(ii) and S. [68, Lemma 4.8] imply that for every $k \in [\![1, p-2]\!]$ there exists (a unique) $\sigma_k \in \Delta_{\mathsf{G}}$ such that for all $\gamma \in \pi_1 S$ it holds

$$\sigma_k\big(\lambda_{\mathsf{G}}(\iota\gamma)\big) = \alpha_k\big(\lambda(\rho\gamma)\big).$$

In particular $\iota : \pi_1 S \to \mathsf{G}$ is $\{\sigma_i\}$-Anosov. Moreover [68, Lemma 4.8] states that $\dim \mathfrak{g}_{\sigma_k} = 1$ for all k.

Theorems 12.3 and 12.1(ii) imply moreover that for every $k \in [\![1, p-2]\!]$ and every $\gamma \in \Gamma$

$$\varpi_k(\lambda_{\mathfrak{so}(p,q)})(\rho\gamma) = n_k \varpi_{\sigma_k}(\lambda_{\mathsf{G}}(\iota\gamma)).$$

It follows that, since $\psi \in \langle\{\varpi_k : k \in [\![1, p-2]\!]\}\rangle$ there exists $\bar{\psi} \in (\mathfrak{a}_{\mathsf{G}})^*$ such that for every $\gamma \in \pi_1 S$ one has

$$\psi(\lambda_{\mathfrak{so}(p,q)}(\rho\gamma)) = \bar{\psi}(\lambda_{\mathsf{G}}(\iota\gamma)).$$

Moreover $\bar{\psi} \in \operatorname{int}(\mathcal{L}_{\{\sigma_i\},\iota})^*$ so we can apply Corollary 11.4 to obtain the desired non-degeneracy. $\square$

We introduce then the following definition.

Definition 12.9. A *length function* on $\mathcal{P}^{p,q}_{\Theta}(S)$ is a map $\psi : \mathcal{P}^{p,q}_{\Theta}(S) \to (\mathfrak{a}_{\Theta})^*$ so that for every $\rho \in \mathcal{P}^{p,q}_{\Theta}(S)$ one has $\psi(\rho) \in \operatorname{int}(\mathcal{L}_{\Theta,\rho})^*$.

Corollary 12.10. *Let $\psi : \mathcal{P}^{p,q}_{\Theta}(S) \to (\mathfrak{a}_{\operatorname{int}\Theta})^*$ be a $\operatorname{Mod}(S)$-invariant length function. Then the semi-definite form $\rho \mapsto \mathbf{P}^{\psi(\rho)}$ induces a $\operatorname{Mod}(S)$-invariant path metric on the space of irreducible positive representations $\mathcal{P}^{p,q}_{\Theta}(S)^{\mathrm{irr}}$.*

We emphasize that our length function has values in $(\mathfrak{a}_{\operatorname{int}\Theta})^*$, a strict subspace of $(\mathfrak{a}_{\Theta})^*$.

Proof. The set of pairs (G, ϕ), where G is a simple Lie group and $\phi : \mathsf{G} \to \mathsf{SO}(p,q)$ is an irreducible representation up to conjugation, is finite. We further restrict the class of such pairs by only considering (G, ϕ) if there exists $\rho \in \mathcal{P}^{p,q}_{\Theta}(S)^{\mathrm{irr}}$ with Zariski closure conjugate to $\phi(\mathsf{G})$. Theorems 12.3 and 12.1(ii) imply then we can stratify $\mathcal{P}^{p,q}_{\Theta}(S)^{\mathrm{irr}}$ by the finitely many submanifolds of

$$W_{(\mathsf{G},\phi)} = \big\{\rho \in \mathcal{P}^{p,q}_{\Theta}(S)^{\mathrm{irr}} : \overline{\rho(\pi_1 S)}^{\mathrm{Z}} \subset \text{ a conjugate of } \phi(\mathsf{G})\big\}.$$

Moreover, by Labourie [49, Theorem 5.2.6] the set

$$W^{\mathrm{Z}}_{(\mathsf{G},\phi)} = \big\{\rho \in W_{(\mathsf{G},\phi)} : \overline{\rho(\pi_1 S)}^{\mathrm{Z}} \text{ is conjugate to } \phi(\mathsf{G})\big\}$$

is open on $W_{(\mathsf{G},\phi)}$ and $W_{(\mathsf{G},\phi)} \backslash W^{\mathrm{Z}}_{(\mathsf{G},\phi)}$ has dimension strictly smaller than $\dim W_{(\mathsf{G},\phi)}$.

Since we have chosen the length function ψ to have values in $(\mathfrak{a}_{\mathrm{int}\,\Theta})^*$, Corollary 12.8(ii) implies that $\rho \mapsto \mathbf{P}^{\psi(\rho)}$ is Riemannian on every $W^{\mathsf{Z}}_{(\mathsf{G},\phi)}$, so Lemma 12.11 below gives the desired conclusion. □

Lemma 12.11 (Bray-Canary-Kao-Martone [12, Lemma 5.2])**.** *Let W_0 be a smooth manifold and let $W_n \subset W_{n-1} \subset \cdots \subset W_1 \subset W_0$ be a nested collection of submanifolds of W_0 so that W_i has non-zero codimension in W_{i-1} for all i. Set $W_{n+1} = \emptyset$. Suppose that g is a smooth non-negative symmetric 2-tensor on W_0 such that for every $i \in [\![0, n]\!]$, the restriction of g to $\mathsf{T}_x W_i$ is positive definite if $x \in W_i \setminus W_{i+1}$. Then, the path pseudo-metric defined by g is a metric.*

Part 3. Hitchin components

Let $\mathfrak{g}$ be a simple split real Lie algebra and $\operatorname{Inn}\mathfrak{g}$ its group of inner automorphisms. Let also $\mathfrak{s} \subset \mathfrak{g}$ be a principal $\mathfrak{sl}_2$ as in § 2.2. This Lie-algebra morphism comes from a Lie-group morphism $\tau_{\mathfrak{g}} = \mathsf{PSL}(2, \mathbb{R}) \to \operatorname{Inn}\mathfrak{g}$ also called *principal*.

Let also S be a closed orientable connected surface of Euler characteristic $\chi(S) < 0$. A representation $\rho : \pi_1 S \to \operatorname{Inn}\mathfrak{g}$ is *Fuchsian* if it factors as

$$\pi_1 S \to \mathsf{PSL}(2, \mathbb{R}) \xrightarrow{\tau_{\mathfrak{g}}} \operatorname{Inn}\mathfrak{g},$$

where the first arrow is discrete and faithful. A connected component of the character variety $\mathfrak{X}(\pi_1 S, \operatorname{Inn}\mathfrak{g})$ that contains a Fuchsian representation will be called a *Hitchin component* of $\operatorname{Inn}\mathfrak{g}$ and denoted by $\mathcal{H}_{\mathfrak{g}}(S)$. Hitchin [37] established that $\mathcal{H}_{\mathfrak{g}}(S)$ is a contractible differentiable manifold of dimension $|\chi(S)| \dim \mathfrak{g}$. The *Fuchsian locus* of Fuchsian representations inside $\mathcal{H}_{\mathfrak{g}}(S)$ is a natural copy of the Teichmüller space of S.

The main purpose of this section is to establish Theorem C, describing degenerations of pressure forms on $\mathcal{H}_{\mathfrak{g}}(S)$. Actually, by Labourie [48] and Beyrer-Labourie-Guichard-Pozzeti-Wienhard [6], Hitchin representations are Δ-Anosov, so representations with Zariski-dense image are already dealt with by Corollary 11.4:

Corollary 12.12. *Let $v \in \mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)$ be non-zero and have Zariski-dense basepoint. Then for every $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$ the set of normalized variations $\mathbb{V}^\psi_v$ has non-empty interior, in particular the pressure form $\mathbf{P}^\psi_\rho$ is Riemannian at ρ.*

This is already enough to establish separation of the path-pseudo metric:

Definition 12.13. A *length function* is a smooth $\operatorname{Mod}(S)$-invariant map $\psi : \mathcal{H}_{\mathfrak{g}}(S) \to \mathfrak{a}^*$ such that for all ρ one has $\psi(\rho) \in \operatorname{int}(\mathcal{L}_\rho)^*$.

Corollary 12.14. *For any length function $\psi : \mathcal{H}_{\mathfrak{g}}(S) \to \mathfrak{a}^*$ the associated pressure semi-norm $\rho \mapsto \mathbf{P}^{\psi(\rho)}$ induces a $\operatorname{Mod}(S)$-invariant path metric on $\mathcal{H}_{\mathfrak{g}}(S)$.*

Proof. The proof works analogous to Corollary 12.10 but easier as the Zariski closures of Hitchin representations are classified by Theorem 13.5 below. Indeed this readily implies that if $\phi : \mathsf{H} \to \mathsf{G}$ is a representation from Table 1 then there is a natural inclusion between the corresponding Hitchin components. Then, Corollary 12.12 gives the needed non-degenerations to apply Lemma 12.11. □

However, understanding degenerations at non-Zariski-dense points is much more subtle and will require some work. This will be finally established in Theorem 15.1, that deals with types A, B, C, D and G_2.

A key object to understand these degenerations is that of *Kostant lines* of $\mathfrak{g}$, by definition these are the 0-restricted weight space of an $\operatorname{ad}\mathfrak{s}$-module.

Kostant lines appear in the statement of Theorem C but are also needed to understand Hausdorff dimension degenerations (§ 17). This is why we will spend some time finding a rather explicit computation of these lines (§ 14). These computations play moreover a role on giving an explicit description of the functional $\varphi \in \mathfrak{a}^*$ whose pressure form $\mathbf{P}^{\varphi}$ is compatible with Goldman's symplectic form at the Fuchsian locus (Corollary 16.4).

13. Necessary facts

The opposition involution i of types A, D and E_6 is induced by a non-trivial external involution $\underline{i} : \operatorname{Inn}\mathfrak{g} \to \operatorname{Inn}\mathfrak{g}$, unique up to conjugation, that induces inturn a non-trivial involution of the character variety that preserves each Hitchin component $\underline{i}^{\mathfrak{X}} : \mathcal{H}_{\mathfrak{g}}(S) \to \mathcal{H}_{\mathfrak{g}}(S)$. We have thus natural inclusions

$$\begin{aligned}
\operatorname{Fix}\underline{i}^{\mathfrak{X}} &= \mathcal{H}_{\mathsf{B}_k}(S) \subset \mathcal{H}_{\mathsf{A}_{2k}}(S),\\
\operatorname{Fix}\underline{i}^{\mathfrak{X}} &= \mathcal{H}_{\mathsf{C}_k}(S) \subset \mathcal{H}_{\mathsf{A}_{2k-1}}(S),\\
\operatorname{Fix}\underline{i}^{\mathfrak{X}} &= \mathcal{H}_{\mathsf{B}_k}(S) \subset \mathcal{H}_{\mathsf{D}_{k+1}}(S),\\
\operatorname{Fix}\underline{i}^{\mathfrak{X}} &= \mathcal{H}_{\mathsf{F}_4}(S) \subset \mathcal{H}_{\mathsf{E}_6}(S).
\end{aligned} \tag{13.1}$$

There is also another natural inclusion $\mathcal{H}_{\mathsf{G}_2}(S) \subset \mathcal{H}_{\mathsf{B}_3}(S)$ given by the fact that the fundamental representation for the short root of $\phi : \mathfrak{G}_2 \to \mathfrak{so}(3,4)$ sends a principal $\mathfrak{sl}_2$ of $\mathfrak{G}_2$ to a principal $\mathfrak{sl}_2$ of $\mathfrak{so}(3,4)$.

Labourie [48] for types A, B, C and G_2 together with Beyrer-Labourie-Pozzetti-Wienhard [6] for a unified approach for all types gives the following (see also Fock-Goncharov [25]).

Theorem 13.1 (Labourie [48])**.** *Every Hitchin representation is* Δ*-Anosov.*

Since Hitchin representations are Δ-Anosov we may consider their *critical hypersurface* as in § 2.13. The following was first stablished by Potrie-S. [60] for types A, B, C and G_2 and the work from Pozzetti-S.-Wienhard [62] together with S. [68] gives a unified approach for all types:

Theorem 13.2 ([60, 62, 68])**.** *For every* $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ *one has* $\Delta \subset \mathcal{Q}_{\rho}$*.*

Convexity of the critical hyper-surface together with the above gives then:

Corollary 13.3 ([60, 62, 68])**.** *If* $\delta \in \mathcal{H}_{\mathfrak{g}}(S)$ *is Fuchsian then every* $\psi \in \operatorname{int}(\mathcal{L}_{\delta})^*$ *has critical entropy at* δ*.*

The following is a consequence of Luzstig's positivity from Fock-Goncharov [25]:

Proposition 13.4. *For every* $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ *and every pair of transverse* $\gamma, h \in \pi_1 S$*, the pair* $\rho(\gamma)$ *and* $\rho(h)$ *is strongly transversally* Δ*-proximal.*

The following recovers a result by Guichard [33] for types A, B, C and G_2.

Theorem 13.5 (S. [68])**.** *Let* $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ *have Zariski closure* H*. Then* $\mathfrak{h}_{ss}$ *is either* $\mathfrak{g}$*, a principal* $\mathfrak{sl}_2(\mathbb{R})$*, or* $\operatorname{Inn}\mathfrak{g}$*-conjugated to one of the possibilities in Table 1.*

We conclude this section with the proof of the following Corollary from the Introduction.

$\mathfrak{g}$	$\mathfrak{h}_{ss}$	$\phi : \mathfrak{h}_{ss} \to \mathfrak{g}$
$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sp}(2n, \mathbb{R})$	defining representation
$\mathfrak{sl}_{2n+1}(\mathbb{R})$	$\mathfrak{so}(n, n+1)\ \forall n$	defining representation
	$\mathfrak{G}_2$ if $n = 3$	fundamental for the short root
$\mathfrak{so}(3, 4)$	$\mathfrak{G}_2$	fundamental for the short root
$\mathfrak{so}(n, n)$	$\mathfrak{so}(n-1, n)\ \forall n \geq 3$	stabilizer of a non-isotropic line
	$\mathfrak{so}(3, 4)$ if $n = 4$	fundamental for the short root
	$\mathfrak{G}_2$ if $n = 4$	stabilizes a non-isotropic line L and is fundamental for the short root on $L^\perp$
$\mathfrak{e}_6$	$\mathfrak{f}_4$	Fix $\underline{i}$ (see Eq. (13.1))

TABLE 1. The statement of Theorem 13.5, if a simple split algebra $\mathfrak{g}$ is not listed in the first column then $\mathfrak{h}_{ss}$ is either $\mathfrak{g}$ or a principal $\mathfrak{sl}_2(\mathbb{R})$; $\mathfrak{e}_6, \mathfrak{f}_4$ and $\mathfrak{G}_2$ denote the split real forms of the corresponding exceptional complex Lie algebras. Observe that there are two non Inn $\mathfrak{so}_{n,n}$-conjugated embeddings $\mathfrak{so}_{n,n-1} \to \mathfrak{so}_{n,n}$ that stabilize a non-isotropic line.

Corollary 13.6 (Curves with arbitrarily small root-variation)**.** *Let $\mathfrak{g}$ be simple split, $\sigma \in \Delta$ and $v \in \mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)$ be non-zero and have Zariski-dense base-point. Then there exists h such that for positive ε and δ there exists $C > 0$ with*

$$\#\big\{[\gamma] \in [\pi_1 S] \text{ primitive} : \varpi_\sigma^\gamma(\rho) \in (t-\varepsilon, t] \text{ and } |\mathrm{d}\sigma^\gamma(v)| \leq \delta\big\} \sim C\frac{e^{ht}}{t^{3/2}}.$$

In particular, for every $\delta > 0$ there exists $\gamma \in \pi_1 S$ such that $|\mathrm{d}\sigma^\gamma(v)| \leq \delta$.

Proof. Since ρ is Hitchin, the flow $\phi^{\varpi_\sigma(\mathcal{J})}$ is Hölder-conjugated to a $\mathrm{C}^{1+\alpha}$-Anosov flow Φ (Potrie-S. [60], Pozzetti-S.-Wienhard [62]). Theorem B implies that group spanned by the periods

$$\Big\{\big(\mathrm{d}\sigma^\gamma(v), \varpi_\sigma^\gamma(\rho)\big) : \gamma \in \pi_1 S\Big\}$$

is dense in $\mathbb{R}^2$. Finally, by Theorem 13.2 $0 \in \operatorname{int} \sigma\big(\mathbb{V}_v^{\varpi_\sigma}\big)$. This places the $\mathrm{C}^{1+\alpha}$-Anosov flow $\Phi = (\Phi_t : \mathsf{U}S \to \mathsf{U}S)_{t\in\mathbb{R}}$ together with the potential $\sigma(\vec{\mathcal{J}})$ in the assumptions of Babillot-Ledrappier [1, Theorem 1.2], where we can pick in their notation $\xi = 0$, thus completing the proof. □

14. Kostant lines

Recall from Kostant [46] that there are rank $\mathfrak{g}$ irreducible adjoint factors of $\mathfrak{s}$ and they have odd dimensions $2e+1$. The numbers e are called *the exponents* of $\mathfrak{g}$ and the associated factor is denoted by V_e. Table 2 gives the exponents for each type.

If e is an exponent of $\mathfrak{g}$, then the 0-restricted-weight space of V_e is a line of $\mathfrak{a}$ that we will denote by $\varkappa^e = \varkappa^e_{\mathfrak{g}} = V_e \cap \mathfrak{a}$ and call *the Kostant line* of exponent e. In this section we task on giving a rather explicit description of these lines.

Remark 14.1. If $e \neq f$ are exponents of $\mathfrak{g}$ then $\varkappa^e$ and $\varkappa^f$ are orthogonal for the Killing form.

Proof. Let V_e^+ be a non-vanishing element of the highest restricted weight space of the associated $\mathfrak{s}$-module. Let also $\mathfrak{s} = \langle E, H, F\rangle$ have the standard relations of an

$\Delta_{\mathfrak{g}}$	exponents
A_d	$1, 2, \ldots, d$
B_d	$1, 3, 5, \ldots, 2d-1$
C_d	$1, 3, 5, \ldots, 2d-1$
D_d	$1, 3, \ldots, 2d-3, d-1$
E_6	$1, 4, 5, 7, 8, 11$
E_7	$1, 5, 7, 9, 11, 13, 17$
E_8	$1, 7, 11, 13, 17, 19, 23, 29$
F_4	$1, 5, 7, 11$
G_2	$1, 5$

TABLE 2. Exponents of irreducible reduced root systems

$\mathfrak{sl}_2$-triple and such that $\mathrm{ad}(E)\cdot V_e^+ = 0$. By definition one has $k_e = \mathrm{ad}(F)^e(V_e^+) \in \varkappa^e$ and is non-zero.

By associativity of the Killing form $(\cdot\,,\cdot)$ one has

$$(k_e, k_f) = (-1)^{e+f}\big((\mathrm{ad}\,F)^e(V_e^+), (\mathrm{ad}\,F)^f(V_f^+)\big) = (-1)^f\big(V_e^+, (\mathrm{ad}\,F)^{e+f}(V_f^+)\big) = 0,$$

since the later are $\mathrm{ad}\,H$-eigenvectors with non-opposite eigenvalues if $f \neq e$. □

Remark 14.2. The longest element of the Weyl group of A_1 acts trivially on the 0-restricted weight space of $\phi_{(2e+1)\varpi_\alpha}$ if e is even and as -1 if e is odd. Consequently, the Kostant lines $\varkappa^e$ are fixed by the longest element of A_d for even exponent e, and are anti-fixed for odd exponent e.

14.1. **A, B and C.** If we denote by $f(x) = x(d-x)$, then the triple $\{E, F, H\}$ below spans a principal $\mathfrak{sl}_2$ of $\mathfrak{sl}_d(\mathbb{R})$, denoted by $\mathfrak{s}$:

$$H = \mathrm{diag}(d-1, d-3, \ldots, 1-d),$$

$$E = \begin{pmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ & & & \end{pmatrix} \text{ and } F = \begin{pmatrix} 0 & & & & \\ f(1) & \ddots & & & \\ & f(2) & \ddots & & \\ & & \ddots & \ddots & \\ & & & f(d-1) & 0 \end{pmatrix}.$$

Consider the matrix product $E^e = E \cdots E$ and define the *Kostant vector*

$$\kappa^e := (-1)^e(\mathrm{ad}\,F)^e(E^e) = \Big[\cdots\big[[E^e, F], F\big]\cdots F\Big].$$

For every $e \in [\![1, d]\!]$ the E^e is annihilated by $\mathrm{ad}\,E$ and is an eigenvector of $\mathrm{ad}\,H$ of eigenvalue $2e$. The span

$$\mathrm{span}\{(\mathrm{ad}\,F)^l \cdot E^e : l \in \mathbb{Z}_{\geq 0}\}$$

is thus an $\mathrm{ad}\,\mathfrak{s}$-module of dimension $2e+1$. The 0-restricted weight space $\mathbb{R}\cdot(\mathrm{ad}\,F)^e(E^e)$ is the Kostant line $\varkappa^e$ and thus $\kappa^e \in \varkappa^e - \{0\}$.

Denote by $\pi^{i,j}$, for $i, j \in [\![1, d]\!]$, the elementary matrix whose only non-vanishing entry is (j, i), and this entry is 1, this is to say, $\pi^{i,j}$ is the operator sending $e_j \mapsto e_i$ and $e_k \mapsto 0$ for every $k \neq j$. Let us also simplify $\pi^{i,i}$ as π^i. Elementary computation gives:

$$[\pi^{i,j}, \pi^{l,t}] = \begin{cases} 0 & \text{if } i \neq t, l \neq j, \\ \pi^{i,t} & \text{if } i \neq t, j = l, \\ \pi^i - \pi^j & \text{if } i = t, l = j, \\ -\pi^{l,j} & \text{if } i = t, l \neq j. \end{cases} \tag{14.1}$$

Also, with this notation one has

$$E^e = \sum_{j=1}^{d-e} \pi^{j,e+j}, \text{ and } F = \sum_{i=1}^{d-1} f(i)\pi^{i+1,i}. \tag{14.2}$$

Proposition 14.3. *One has that*

$$\begin{aligned} \mathsf{K}^e &= (-1)^e \sum_{l=1}^{d-e} \Big(f(l) \cdots f(l+e-1) \cdot \sum_{t=0}^{e} (-1)^t \binom{e}{t} \pi^{l+e-t} \Big) \\ &= \sum_{j=1}^{d} \pi^j \Big(\sum_{t=0}^{e} (-1)^t \binom{e}{t} f(j-t) \cdots f(j-t+e-1) \Big). \end{aligned} \tag{14.3}$$

For example one has

$$\begin{aligned} \mathsf{K}^2 &= 2 \sum_{j=1}^{d} \big(d^2 + 3d(1-2j) + 6j(j-1) + 2 \big) \pi^j, \\ \mathsf{K}^3 &= 6 \sum_{j=1}^{d} (-2j+1+d)(d^2 - 10dj + 10j^2 + 5d - 10j + 6)\pi^j, \\ \mathsf{K}^{d-1} &= f(1) \cdots f(d-1) \sum_{j=1}^{d} (-1)^{j-1} \binom{d-1}{d-j} \pi^j. \end{aligned} \tag{14.4}$$

Proof. We compute $(\operatorname{ad} F)^e(E^e)$. Using Equation (14.2) this translates to computing the brackets

$$\Big[\sum_{i=1}^{d-1} f(i)\pi^{i+1,i}, \big[\cdots, \big[\sum_{i=1}^{d-1} f(i)\pi^{i+1,i}, \sum_{j=1}^{d-e} \pi^{j,e+j} \big] \cdots \big] \Big],$$

for which we use the elementary computations in Equation (14.1). To do so, we use a recursive argument, for which we compute the bracket $[F, a(l,t)\pi^{l,t}]$ for arbitrary $l, t \in [\![1, d]\!]$ and some real-valued function a. Direct computation gives then

$$[F, a(l,t)\pi^{l,t}] = a(l,t)\big(f(l)\pi^{l+1,t} - f(t-1)\pi^{l,t-1} \big).$$

Applying again $[F, \cdot]$, the term $\pi^{l+1,t-1}$ will appear once for each factor $\pi^{l+1,t}$ and $\pi^{l,t-1}$, with coefficient $f(l)f(t-1)(-1)2$. If one further applies $[F, \cdot]$ one readily sees the binomial coefficients with the alternating signs appearing as the coefficient of $\pi^{l+k-j,t-j}$ (together with the corresponding f's), this is to say

$$\begin{aligned} (\operatorname{ad} F)^k(\pi^{l,t}) &= f(l) \cdots f(l+k-1)\pi^{l+k,t} \\ &\quad + \sum_{i=1}^{k-1} f(l) \cdots f(l+k-i-1) \cdot f(t-1) \cdots f(t-i)(-1)^i \binom{k}{i} \pi^{l+k-i,t-i} \\ &\quad + (-1)^k f(t-1) \cdots f(t-k)\pi^{l,t-k} \end{aligned} \tag{14.5}$$

Thus, replacing $t = l + e$ and $k = e$ one has:

$$\begin{aligned}(\operatorname{ad} F)^e(\pi^{l,l+e}) &= f(l)\cdots f(l+e-1)\pi^{l+e,l+e}\\ &+ \sum_{i=1}^{e-1} f(l)\cdots f(l+e-i-1)\cdot f(l+e-1)\cdots f(l+e-i)(-1)^i\binom{e}{i}\pi^{l+e-i,l+e-i}\\ &+ (-1)^e f(l+e-1)\cdots f(l+e-e)\pi^{l,l+e-e}\\ &= f(l)\cdots f(l+e-1)\sum_{i=0}^{e}(-1)^i\binom{e}{i}\pi^{l+e-i}. \qquad (14.6)\end{aligned}$$

Summing on l from 1 to $d-e$ gives the first required formula.

The second equality is not completely immediate from the first so we quickly explain how it is obtained. By standard reordering of the sum one gets:

$$\begin{aligned}\mathsf{K}^e &= (-1)^e\Big(\sum_{l=1}^{d-e} f(l)\cdots f(l+e-1)\cdot\sum_{t=0}^{e}(-1)^t\binom{e}{t}\pi^{l+e-t}\Big)\\ &= (-1)^e\sum_{s=0}^{e}(-1)^{e-s}\sum_{j=1}^{d-e}\big(\binom{e}{s}f(j)\cdots f(j+e-1)\big)\pi^{j+s} \qquad (s=e-t)\\ &= \sum_{s=0}^{e}(-1)^s\sum_{i=s+1}^{d-e+s}\big(\binom{e}{s}f(i-s)\cdots f(i-s+e-1)\big)\pi^{i} \qquad (i=j+s).\end{aligned}$$

One observes then that for every $i \in [\![1,s]\!]$ the number $f(i-s)\cdots f(i-s+e-1) = 0$, since $i-s \leq 0$ and $i-s+e-1 \geq 0$ (recall $s \in [\![0,e]\!]$), so one can extend the lower index of the sum in i in the above formula to starting from $i=1$ and the sum will be unchanged. Analogous reasoning allows to extend the upper index of the sum (recall $f(x) = f(d-x)$) so the proof is complete. □

We will use the following computation to describe the adjoint factors in the Hitchin component.

Lemma 14.4 (Exponents are shifted)**.** *Consider $e,k \in [\![2,d-1]\!]$ then the vector $[[F,E^e],E^k] \in \mathbb{R}\cdot E^{e+k-1}$. Moreover, for $k \leq d-3$ the vector $[[F,E^3],E^k]$ is non-zero.*

Proof. The centralizer of E has dimension $d-1$ (Kostant [46, Corollary 5.3]). It is thus spanned, as a vector space, by $\{E^l : l \in [\![1,d-1]\!]\}$. The first assertion of the lemma follows by the combination of two straightforward calculations:

$$\begin{aligned}\operatorname{ad}_E\big([[F,E^e],E^k]\big) &= [\operatorname{ad}_E([F,E^e]),E^k] = \big[[H,E^e],E^k\big] = 2e[E^e,E^k] = 0;\\ \operatorname{ad}_H\big([[F,E^e],E^k]\big) &= \big[\operatorname{ad}_H([F,E^e]),E^k\big] + \big[[F,E^e],\operatorname{ad}_H(E^k)\big]\\ &= (2e-2+2k)\big[[F,E^e],E^k\big].\end{aligned}$$

Indeed, the first computation gives that the desired element belongs to the span of $\{E^l : l \in [\![1,d-1]\!]\}$, and the second asserts that it is an eigenvector of ad_H of eigenvalue $2(e+k-1)$, giving the desired conclusion.

To show that $[[F,E^3],E^k]\neq 0$ if $k\leq d-3$ we use the formulae from Equation (14.1). One has

$$[F,E^3]=\sum_{j=1}^{d-3} f(j)\pi^{j+1,j+3}-f(j+2)\pi^{j,j+2}.$$

Since we intend to further bracket with $E^k=\sum_{l=1}^{d-k}\pi^{l,k+l}$ we observe that

$$[\pi^{j+1,j+3},\pi^{l,k+l}]=\begin{cases}\pi^{j+1,k+j+3} & \text{if } l=j+3,\\ -\pi^{j+1-k,j+3} & \text{if } k+l=j+1,\\ 0 & \text{otherwise,}\end{cases}$$

where both non-vanishing options cannot simultaneously occur (since $k\neq -2$). Similarly one has

$$[\pi^{j,j+2},\pi^{l,k+l}]=\begin{cases}\pi^{j,j+k+2} & \text{if } l=l+2,\\ -\pi^{j-k,j+2} & \text{if } k+l=j,\\ 0 & \text{otherwise.}\end{cases}$$

Putting together the last three equations, one has

$$\big[[F,E^3],E^k\big]=\sum_{j=1}^{d-3} f(j)\big(\pi^{j+1,k+j-3}-\pi^{j+1-k,j+3}\big)-f(j+2)\big(\pi^{j,j+k+2}-\pi^{j-k,j+2}\big).$$

We show then that the coefficient of $[[F,E^3],E^k]$ in the element $\pi^{1,k+3}$ is non-zero. Indeed this coefficient is

$$-f(3)+f(k+3)-f(k)=-6k\neq 0$$

if $k\leq d-4$ or $-2f(3)\neq 0$ if $k=d-3$. □

Proposition 14.5 (Adjoint Factors)**.**

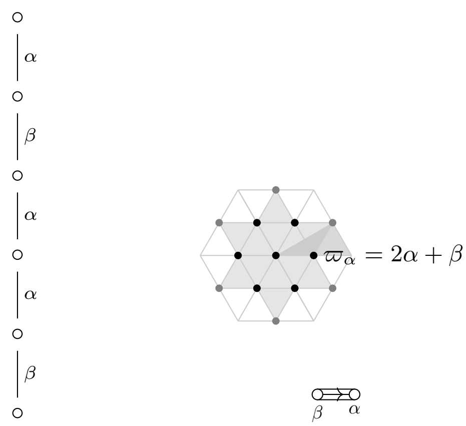
- *Let $\mathfrak{g}=\mathfrak{so}(n,n+1)$ or $\mathfrak{sp}(2n,\mathbb{R})$ and $\phi:\mathfrak{g}\to\mathfrak{sl}(d,\mathbb{R})$ be the defining representation. Then as an $\mathrm{ad}(\phi\,\mathfrak{g})$-module one has*

  $$\mathfrak{sl}(d,\mathbb{R})=\phi_{2\varpi_{\alpha_1}}\oplus\phi_{\varpi_{\alpha_2}}.$$

  These two factors also correspond to the decomposition of $\mathfrak{sl}(d,\mathbb{R})$ in odd versus even exponents.
- *Let now $\mathfrak{G}_2$ be a real-split form of the exceptional complex Lie algebra of type G_2 and let $\phi:\mathfrak{G}_2\to\mathfrak{sl}_7(\mathbb{R})$ be the fundamental representation associated to the short root. Then $\phi(\mathfrak{G}_2)$ has three adjoint factors given by $V_1\oplus V_5$, V_3 and $V_2\oplus V_4\oplus V_6$.*

Proof. We focus on the first item, the second following similarly but with more involved computations that we omit. Using the computation for the exponents of $\mathfrak{g}$ in Table 2 one sees that $\phi(\mathfrak{g})=\sum_{\text{odd } e}V_e$ is an irreducible factor. Moreover, as $[F,E^3]\in\phi(\mathfrak{g})$, Lemma 14.4 implies that all even exponents belong to the same irreducible factor, giving the result. □

Corollary 14.6. *Let $\mathfrak{g}$ be either $\mathfrak{sp}(2n,\mathbb{R})$, $\mathfrak{so}(n,n+1)$ or $\mathfrak{G}_2$, and denote by $\phi_0:\mathfrak{g}\to\mathfrak{sl}(d,\mathbb{R})$ be either the defining representation in the first two cases or the fundamental representation for the short root in the last case and let $\boldsymbol{\phi}:\mathfrak{g}\to V_{\mathfrak{g}}$ be an adjoint factor of $\phi_0(\mathfrak{g})$, then $\theta=\Delta$ and for any $X_0\in\mathrm{Fix}\,\mathrm{i}\subset\mathfrak{a}^+$ the cone of $(\boldsymbol{\phi},X_0)$-compatible elements is $\mathcal{X}_\phi=\mathfrak{a}^+$.*



$$\Delta^+ = \{\alpha, \beta, \alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta\}$$

FIGURE 4. Hasse diagram for the 7-dimensional irreducible representation of G_2, which is the fundamental representation of the short root, together with the corresponding weight sets (in black). Recall also that $\varpi_\beta = 3\alpha + 2\beta$.

Proof. In the first two cases the result follows readily as, by direct computation, the weights of $\mathrm{ad}(\phi)$ are integer multiples of simple roots of $\mathfrak{g}$ and every root appears. The $\mathfrak{G}_2$ case follows by explicit verification. See Figure 4. ☐

14.2. **Cleaner formulae for $\varkappa^e$.** We proceed to a more explicit computation of κ^e and notably of $\sigma_j(\kappa^e)$. To this end, let us consider the classical *(finite) difference operator* defined, for a function $g : \mathbb{R} \to \mathbb{R}$, by

$$\triangle g(x) = g(x+1) - g(x).$$

We also consider, for a real number $z \in \mathbb{R}$ (a slight modification of) the *falling factorial* notation: for $k \in \mathbb{N}$ we let

$$z^{\underline{k}} = z(z-1)\cdots(z-k+1),$$

with the convention that $z^{\underline{0}} = 1$, in particular $0^{\underline{0}} = 1$, and for later use we define $z^{\underline{-k}} = 0$. For a function g, we let $g(x)^{\underline{k}}$ be the k-th falling factorial applied to the real number $g(x)$.

Straightforward computations yield the following rules.

Lemma 14.7. *For every $k \in \mathbb{N}$ one has*

i) $\triangle^k g(x) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} g(x+k-i)$;

ii) a Leibnitz rule $\triangle^k (gh)(x) = \sum_{i=0}^{k} \binom{k}{i} \triangle^i g(x) \triangle^{k-i} h(x+i)$;

iii) $\triangle x^{\underline{k}} = k x^{\underline{k-1}}$ *and if we let $r(x) = l - x$ for some $l \in \mathbb{R}$, then*

$$\triangle\big(r(x)\big)^{\underline{k}} = -k r(x+1)^{\underline{k-1}}.$$

Considering the function $g_{d,e}(x) = d + e - x$, together with

$$F_{d,e}(x) = f(x-e)f(x-e+1)\cdots f(x-1) = (x-1)^{\underline{e}} \cdot \big(g_{d,e}(x)\big)^{\underline{e}}, \tag{14.7}$$

Proposition 14.3 yields

$$\kappa^e = \sum_{j=1}^{d} \pi^j \Big(\sum_{t=0}^{e} (-1)^t \binom{e}{t} F_{d,e}(j-t+e) \Big) = \sum_{j=1}^{d} \pi^j \triangle^e F_{d,e}(j),$$

where the last equality comes from Lemma 14.7. We compute then $\triangle^e F_{d,e}(x)$ using the Leibnitz rule applied to the product $F_{d,e}(x) = (x-1)^{\underline{e}} \cdot (g_{d,e}(x))^{\underline{e}}$.

$$\begin{aligned}
\triangle^e F_{d,e}(x) &= \sum_{i=0}^{e} \binom{e}{i} \triangle^i (x-1)^{\underline{e}} \triangle^{e-i} (g_{d,e}(x+i))^{\underline{e}} \\
&= \sum_{i=0}^{e} (-1)^{e-i} \binom{e}{i} \frac{e!}{(e-i)!} (x-1)^{\underline{e-i}} \frac{e!}{i!} \big(g_{d,e}(x+e)\big)^{\underline{i}} \\
&= (-1)^e e! \sum_{i=0}^{e} (-1)^i \binom{e}{i}^2 (x-1)^{\underline{e-i}} \big(g_{d,e}(x+e)\big)^{\underline{i}}.
\end{aligned}$$

In order to decide whether κ^e belongs to the kernel of a simple root $\sigma_j \in \Delta$ we compute $-\sigma_j(\kappa^e) = \triangle^{e+1} F_{d,e}(j)$, which we write, by Lemma 14.7, as

$$\begin{aligned}
\triangle^{e+1} F_{d,e}(x) &= \sum_{i=0}^{e+1} \binom{e+1}{i} \triangle^i (x-1)^{\underline{e}} \triangle^{e+1-i} (g_{d,e}(x+i))^{\underline{e}} \\
&= \sum_{i=0}^{e+1} (-1)^{e+1-i} \binom{e+1}{i} \frac{e!}{(e-i)!} (x-1)^{\underline{e-i}} \frac{e!}{i-1!} \big(g_{d,e}(x+e+1)\big)^{\underline{i-1}} \\
&= (-1)^{e+1}(e+1)! \sum_{i=1}^{e} (-1)^i \binom{e}{i} \binom{e}{i-1} (x-1)^{\underline{e-i}} (d-1-x)^{\underline{i-1}}.
\end{aligned}$$

We record the above computations in the following lemma.

Lemma 14.8. *One has*

$$\kappa^e = (-1)^e e! \sum_{j=1}^{d} \pi^j \Big(\sum_{t=0}^{e} (-1)^t \binom{e}{t}^2 (j-1)^{\underline{e-t}} (d-j)^{\underline{t}} \Big);$$

$$\sigma_j(\kappa^e) = (-1)^e (e+1)! \sum_{t=1}^{e} (-1)^t \binom{e}{t} \binom{e}{t-1} (j-1)^{\underline{e-t}} (d-1-j)^{\underline{t-1}}. \tag{14.8}$$

Remark 14.9. In particular one has $\varpi_1(\kappa^e) = e!(d-1)^{\underline{e}} > 0$.

14.3. **Type D.** Consider a $2n$ dimensional real vector space equipped with a bilinear form ω of signature (n,n) and let $\mathsf{SO}_{n,n}$ be the volume preserving automorphisms of this form. Let also $x \mapsto x^*$ be the adjoint operator on $\mathfrak{sl}(2n,\mathbb{R})$ defined by ω. One has

$$\mathfrak{so}(n,n) = \{x \in \mathfrak{sl}(2n,\mathbb{R}) : x + x^* = 0\}.$$

Consider a non-isotropic line ℓ and its orthogonal complement $\ell^\perp$ for ω.

	$\mathfrak{sl}_3(\mathbb{R})$	$\mathfrak{sl}_4(\mathbb{R})$	$\mathfrak{sl}_5(\mathbb{R})$	$\mathfrak{sl}_6(\mathbb{R})$
κ^1	$(2,0,-2)$	$(3,1,-1,-3)$	$(4,2,0,-2,-4)$	$(5,3,1,-1,-3,-5)$
κ^2	$(4,-8,4)$	$12\cdot(1,-1,-1,1)$	$12\cdot(2,-1,-2,-1,2)$	$8\cdot(5,-1,-4,-4,-1,5)$
κ^3		$36\cdot(1,-3,3,-1)$	$144\cdot(1,-2,0,2,-1)$	$72\cdot(5,-7,-4,4,7,-5)$
κ^4			$576\cdot(1,-4,6,-4,1)$	$2880\cdot(1,-3,2,2,-3,1)$
κ^5				$14400\cdot(1,-5,10,-10,5,-1)$

	$\mathfrak{sl}_7(\mathbb{R})$	$\mathfrak{sl}_8(\mathbb{R})$
κ^1	$(6,4,2,0,-2,-4,-6)$	$(7,5,3,1,-1,-3,-5,-7)$
κ^2	$12\cdot(5,0,-3,-4,-3,0,5)$	$12\cdot(7,1,-3,-5,-5,-3,1,7)$
κ^3	$720\cdot(1,-1,-1,0,1,1,-1)$	$180\cdot(7,-5,-7,-3,3,7,5,-7)$
κ^4	$2880\cdot(3,-7,1,6,1,-7,3)$	$2880\cdot(7,-13,-3,9,9,-3,-13,7)$
κ^5	$86400\cdot(1,-4,5,0,-5,4,-1)$	$43200\cdot(7,-23,17,15,-15,-17,23,-7)$
κ^6	$518400\cdot(1,-6,15,-20,15,-6,1)$	$3628800\cdot(1,-5,9,-5,-5,9,-5,1)$
κ^7		$25401600\cdot(1,-7,21,-35,35,-21,7,-1)$

TABLE 3. The Kostant vectors of $\mathfrak{sl}_d(\mathbb{R})$ for $d\in[\![3,8]\!]$.

We have then an ω-preserving involution i with $i|\ell=-\mathrm{id}$ and $i|\ell^{\perp}=\mathrm{id}$, which gives an involution $\underline{i}:\mathsf{SO}_{n,n}\to\mathsf{SO}_{n,n}$ defined by $g\mapsto igi$. The group of fixed points of $\underline{i}$ is the subgroup of $\mathsf{SO}_{n,n}$ that stabilizes ℓ. For $g\in\operatorname{Fix}\underline{i}$, the restriction $g\mapsto g|\ell^{\perp}$ gives an isomorphism of $(\operatorname{Fix}\underline{i})_0$ with a special orthogonal group of signature $(n-1,n)$.

The differential $d_e\underline{i}:\mathfrak{so}_{n,n}\to\mathfrak{so}_{n,n}$ coincides with $x\mapsto ixi$ and is a Lie-algebra involution giving a decomposition

$$\mathfrak{so}_{n,n}=\operatorname{Lie}(\operatorname{Fix}\underline{i})\oplus\{x\in\mathfrak{so}_{n,n}:\mathrm{d}_e\underline{i}(x)=-x\}. \tag{14.9}$$

If $x\in\mathfrak{sl}(2n,\mathbb{R})$ is anti-fixed by $d_e\underline{i}$ then one readily observes that $x(\ell)\subset\ell^{\perp}$ and $x(\ell^{\perp})\subset\ell$. We can easily describe then the anti-fixed subspace of $\mathfrak{so}(n,n)$ as

$$\{x\in\mathfrak{so}_{n,n}:\mathrm{d}_e\underline{i}(x)=-x\}=\left\{x-x^*:x\in\hom(\ell,\ell^{\perp})\right\}.$$

It is a $2n-1$-dimensional vector space and an irreducible $\operatorname{Lie}(\operatorname{Fix}\underline{i})$-module.

If $\mathfrak{s}$ is a principal $\mathfrak{sl}_2$ of $\mathfrak{so}_{n,n}$ then it stabilizes a non-isotropic line, which we can assume to be ℓ, and acts irreducibly on $\ell^{\perp}$, so we conclude that

$$V_{n-1,\mathrm{a}}:=\{x:\mathrm{d}_e\underline{i}(x)=-x\}$$

is an irreducible $\mathfrak{s}$-factor of dimension $2(n-1)+1$.

The a in the notation solves an ambiguity issue when n is even. Indeed, observe from Table 2 that two situations occur for $\mathfrak{so}_{n,n}$. If n is odd, the exponent $n-1$ occurs with multiplicity one and is the only even exponent of $\mathfrak{so}_{n,n}$. However if n is even, there are two $\mathfrak{s}$-adjoint factors of dimension $2(n-1)+1$. One of these factors is contained in $\operatorname{Fix}\mathrm{d}_e\underline{i}$, and the other one is $V_{n-1,\mathrm{a}}$.

Let $\mathfrak{a}_{\mathfrak{so}_{n,n}}=\mathbb{R}^n$ be a Cartan subspace of $\mathfrak{so}_{n,n}$ and consider the set of simple roots $\Delta=\{\sigma_1,\dots,\sigma_{n-1},\alpha_n\}$ where $\sigma_i(a)=a_i-a_{i+1}$ and $\alpha_n(a)=a_{n-1}+a_n$. We can choose a Cartan subspace $\mathfrak{a}_{\mathfrak{so}_{n-1,n}}$ of $\operatorname{Fix}\underline{i}$ that is embedded in $\mathfrak{a}_{\mathfrak{so}_{n,n}}$ as

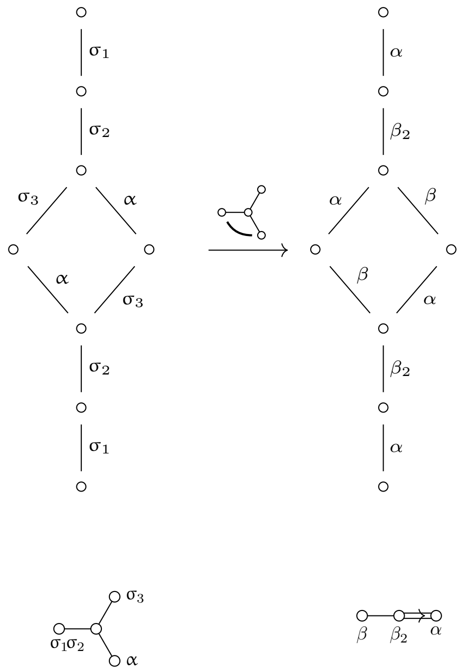


FIGURE 5. The irreducible representation $\mathfrak{so}(3,4) \to \mathfrak{so}(4,4)$.

$\{a \in \mathbb{R}^n : a_n = 0\}$. The involution $\mathrm{d}_e \underline{i}$ acts on $\mathfrak{a}_{\mathfrak{so}_{n,n}}$ as

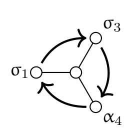
$$\mathrm{i} := (a_1, \ldots, a_n) \mapsto (a_1, \ldots, -a_n).$$

Observe that this involution sends σ_{n-1} to α_n and fixes the other roots so it's the opposition involution i of $\mathfrak{a}_{\mathfrak{so}_{n,n}}$. Moreover, the Kostant line associated to the anti-fixed factor is

$$\varkappa^{n-1,\mathrm{a}} := \mathbb{R} \cdot (0, \ldots, 0, 1).$$

The remaining Kostant lines are the ones of $\mathfrak{so}_{n-1,n}$ embedded in $\mathfrak{a}_{\mathfrak{so}_{n,n}}$ via the above inclusion.

14.3.1. *Triality.* We now deal with the special case D_4. In this special case the Dynkin diagram has an order three automorphism τ that fixes σ_2 and $\sigma_1 \mapsto \sigma_3$, $\sigma_3 \mapsto \alpha_4$ and $\alpha_4 \mapsto \sigma_1$, see Equation (14.10).



(14.10)

This automorphism can be realized as the orthogonal transformation

$$\tau := \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix} : \mathfrak{a}_{\mathfrak{so}_{4,4}} \to \mathfrak{a}_{\mathfrak{so}_{4,4}}.$$

This automorphism can also be extended to an external automorphism of $\underline{\tau}$: $\mathfrak{so}_{4,4} \to \mathfrak{so}_{4,4}$ whose fixed point set $\operatorname{Fix} \tau = \phi_{\varpi_\alpha}(\mathfrak{G}_2)$. Moreover, the involution

$\tau\,\mathrm{i}\,\tau^{-1}$ of $\mathfrak{a}_{\mathfrak{so}_{4,4}}$ has fixed-point set the image $\phi_\alpha(\mathfrak{so}_{3,4})$ of the fundamental representation for the short root of $\mathfrak{so}_{3,4}$, see Figure 5. The adjoint factors of $\phi_\alpha(\mathfrak{so}_{3,4})$ are then

$$\mathfrak{so}_{4,4} = \big(\underline{\tau}(V_1 \oplus V_3 \oplus V_5)\big) \oplus (\underline{\tau}(V_{3,\mathrm{a}})).$$

Using the explicit formula for $\tau : \mathfrak{a}_{\mathfrak{so}_{4,4}} \to \mathfrak{a}_{\mathfrak{so}_{4,4}}$ one has

$$\underline{\tau}(V_{3,\mathrm{a}}) \cap \mathfrak{a}_{\mathfrak{so}_{4,4}} = \tau(\varkappa^{3,\mathrm{a}}) = \mathbb{R} \cdot (-1, 1, 1, -1). \tag{14.11}$$

This last equation will be needed in the proof of Theorem 15.1.

15. Pressure degenerations are Lie-theoretic

In this section we establish the following. Recall that if $\delta \in \mathcal{H}_g(S)$ is Fuchsian, we havce denoted by

$$\mathsf{T}^e_\delta = H^1_{\mathrm{Ad}\,\delta}(\pi_1 S, V_e).$$

Theorem 15.1. *Let $\mathfrak{g}$ be simple split of type A, B, C, D or G_2. Consider $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ and a length functional $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ then, the pressure form $\mathbf{P}^\psi_\rho$ is degenerate at $v \in \mathsf{T}_\rho\mathcal{H}_{\mathfrak{g}}(S)$ if and only if either of the following situations hold:*

- *ρ is Fuchsian and*

$$v \in \bigoplus_{e:\psi(\varkappa^e)=0} \mathsf{T}^e_\delta,$$

- *ρ is self dual, ψ is i-invariant and v is $\bar{\mathrm{i}}$-anti-invariant.*
- *$\mathfrak{g}$ is of type A_6 or C_3, the Zariski closure of $\rho(\pi_1 S)$ is $\phi_{\varpi_\alpha}(\mathfrak{G}_2)$, $\psi(\varkappa^3) = 0$ and $v \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3)$.*
- *$\mathfrak{g}$ is of type D_4, the Zariski closure of $\rho(\pi_1 S)$ is conjugate to $\phi_\alpha(\mathsf{SO}_{3,4})$, $v \in H^1_{\mathrm{Ad}\,\rho}\big(\pi_1 S, \underline{\tau}(V_{3,\mathrm{a}})\big)$ and $\psi(-1, 1, 1, -1) = 0$.*

We begin the proof of Theorem 15.1 with some preparation lemmas.

15.1. Preparation Lemmas of independent interest I.

Lemma 15.2. *Let $\delta \in \mathcal{H}_{\mathfrak{g}}(S)$ be Fuchsian and e an exponent of $\mathfrak{g}$. For any non-zero $\mathsf{u} \in \mathsf{T}^e_\delta$ and every $\psi \in \mathrm{int}\,(\mathcal{L}_\delta)^*$ such that $\psi(\varkappa^e) \neq 0$ the set of normalized variations $\mathbb{V}^\psi_{\mathsf{u}} \subset \varkappa^e$ has non-empty interior.*

Proof. Corollary 13.3 and Equation (2.19) imply that

$$0 = \partial^{\log} \hbar^\psi = \psi(p_\psi \mathsf{u}) \in \psi\big(\mathbb{V}^\psi_{\mathsf{u}}\big).$$

Since $\ker\psi \cap \varkappa^e = \{0\}$ and $\mathbb{V}^\psi_{\mathsf{u}} \subset \varkappa^e$ by Corollary 8.6, the above equation yields $0 \in \mathbb{V}^\psi_{\mathsf{u}}$. However, since $\mathbb{V}^\psi_{\mathsf{u}}$ is convex, if its interior where empty then it would be reduced to the point $\{0\} = \mathbb{V}^\psi_{\mathsf{u}}$. This yields that, for every $\gamma \in \pi_1 S$ one has

$$\mathrm{d}\lambda^\gamma(\mathsf{u}) = 0,$$

in particular that $\mathrm{d}_\rho \lambda^\gamma_1(\mathsf{u}) = 0$ for all γ, contradicting [14, Proposition 10.1] with the fact that $\mathsf{u} \neq 0$. $\square$

Lemma 15.3. *Let $\rho \in \mathcal{H}_{\mathsf{A}_{d-1}}(S)$ and $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$.*

(i) *If $\rho(\pi_1 S)$ has Zariski-closure $\mathsf{SO}(n,n+1)$ or $\mathsf{PSp}(2n,\mathbb{R})$ according to the parity if d, or Zariski-closure $\phi_{\varpi_\alpha}(\mathsf{G}_2)$ if $d=7$, and if we let $\mathfrak{g}$ be the corresponding Lie algebra, then*

$$V_{\mathfrak{g}} = \bigoplus_{e\ \mathit{even}} V_e$$

is an adjoint factor of ρ and for every non-zero $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_{\mathfrak{g}})$

$$\mathbb{V}^{\psi}_{\mathsf{u}} \subset V_{\mathfrak{g}}$$

has non-empty interior.

(ii) *If $d=7$ and the Zariski closure of $\rho(\pi_1 S)$ is $\phi_{\varpi_\alpha}(\mathsf{G}_2)$, then for any non-trivial cocycle $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3)$ the set $\mathbb{V}^{\psi}_{\mathsf{u}} \subset \varkappa^3$ has non-empty interior and contains $\{0\}$ in its interior. In particular $\mathbf{P}^{\psi}(\mathsf{u}) \neq 0$.*

Proof. By Remark 14.2 w_0 acts trivially on even exponent spaces, so the first item is a consequence of Corollary 14.6 together with Remark 7.1. To deal with the second item, we restrict ourselves to $\phi_{\varpi_\alpha}(\mathsf{G}_2) < \mathsf{SO}(3,4)$ with Cartan subspace $\mathfrak{a}_{\mathfrak{so}(3,4)} = \mathbb{R}^3$, Weyl chamber $\{a \in \mathbb{R}^3 : a_1 \geq a_2 \geq a_3 \geq 0\}$ and simple roots

$$\sigma_1(a) = a_1 - a_2,\ \ \sigma_2(a) = a_2 - a_3 \text{ and } \varepsilon_3(a) = a_3.$$

The subalgebra $\phi_{\varpi_\alpha}(\mathfrak{G}_2)$ has Cartan subspace $\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)} \subset \mathfrak{a}_{\mathfrak{so}(3,4)}$ given by

$$\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)} = \{(a_1, a_2, a_1 - a_2) : a_1, a_2 \in \mathbb{R}\}$$

and simple roots $\{\sigma_1, \sigma_2\}$, see Figure 4.

Since $\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)} = \ker(\varepsilon - \sigma_1)$, any convex combination $t\varepsilon - (1-t)\sigma_1$ coincides with σ_1 when restricted to $\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)}$. Since $\hbar^{\sigma_1}_{\rho} = 1$ by Theorem 13.2, it follows that the affine line $\{t\varepsilon + (1-t)\sigma_1 : t \in \mathbb{R}\}$ is contained in the critical hypersurface $\mathcal{Q}_\rho$ of $\rho : \pi_1 \to \phi_{\varpi_\alpha}(\mathsf{G}_2) < \mathsf{SO}(3,4)$, see Figure 6. Fix such a combination, $\varphi = (1/2)(\varepsilon + \sigma_1)$ for example and observe, from Table 3 that $\varkappa^3 = \mathbb{R}\cdot(1,-1,-1)$ and that $\varphi(1,-1,-1) = 1/2 \neq 0$.

If we let v be the tangent vector associated to $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3)$, and ρ_t be a short curve tangent to v, then again Theorem 13.2 implies that $\{\varepsilon, \sigma_1\} \subset \mathcal{Q}_{\rho_t}$ giving that the entropy $\hbar^{\varphi}_{\rho_t}$ is critical at ρ. We are now in the exact situation of Lemma 15.4(ii) (with the roles of φ and ψ reversed). We obtain that the set of variations $\mathbb{V}^{\psi}_{\mathsf{u}} \subset \varkappa^3$ has non-empty interior as desired. The last statement now follows from Lemma 15.4(iii). □

15.2. **Pressure forms: some information on lower strata.** We now focus on Δ-Anosov representations of a finitely generated group Γ.

Let $\rho : \Gamma \to \mathsf{G}$ be a such a representation and consider a cocycle $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\Gamma, \mathfrak{g})$ that we assume integrable, so we also denote by $\mathsf{u} \in \mathsf{T}_\rho \mathfrak{X}(\Gamma, \mathsf{G})$ the associated tangent vector.

Lemma 15.4. *Let $\rho \in \mathfrak{A}_{\Delta}(\Gamma, \mathsf{G})$ have semi-simple Zariski closure H. Fix $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ and a disjoined adjoint factor V_{H} of H. Consider an integrable cocycle $\mathsf{u} \in H^1_{\mathrm{Ad}\,\rho}(\Gamma, V_{\mathsf{H}})$ such that there exists $\gamma \in \Gamma$ with $\mathrm{d}\lambda^{\gamma}(\mathsf{u}) \neq 0$, then:*

(i) *If $V_{\mathsf{H}} \cap \mathfrak{a} \subset \ker\psi$, then $\mathbf{P}^{\psi}$ degenerates at u.*

(ii) *If $V_{\mathsf{H}} \cap \mathfrak{a} \cap \ker\psi = \{0\}$ and $\mathrm{d}\hbar^{\psi}(\mathsf{u}) = 0$ then for every $\varphi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ with $V_{\mathsf{H}} \cap \mathfrak{a} \cap \ker\varphi = \{0\}$ the set $\mathbb{V}^{\varphi}_{\mathsf{u}}$ has non-empty interior and $0 \in \mathrm{int}\,\mathbb{V}^{\varphi}_{\mathsf{u}}$.*

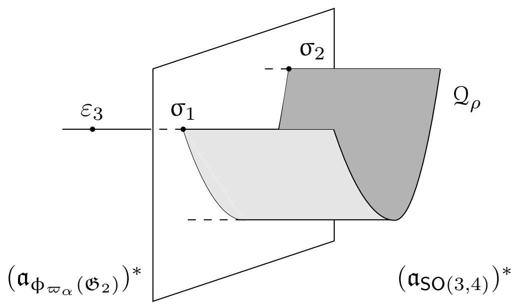


FIGURE 6. The critical hypersurface in $\mathfrak{a}^*_{\mathsf{SO}(3,4)}$ of a Hitchin representation ρ whose Zariski closure is $\phi_{\varpi_\alpha}(\mathsf{G}_2)$.

(iii) *If* $V_{\mathsf{H}} \cap \mathfrak{a} \cap \ker\psi = \{0\}$ *and* $\mathbf{P}^\psi(\mathsf{u}) = 0$ *then* $\mathbb{V}^\psi_{\mathsf{u}}$ *is reduced to a point and is non-zero.*

(iv) *Assume* H *has rank* 1. *If* $\psi, \varphi \in \operatorname{int}(\mathcal{L}_\rho)^*$ *both have kernel whose intersection with* $V_{\mathsf{H}} \cap \mathfrak{a}$ *vanishes, then there exists* $c > 0$ *such that for all* $\mathsf{v} \in H^1_{\operatorname{Ad}\rho}(\Gamma, V_{\mathsf{H}})$ *one has*

$$\mathbf{P}^\psi_\rho(\mathsf{v}) = c\mathbf{P}^\varphi_\rho(\mathsf{v}).$$

Proof. If the derivative of $(\rho_t) \in \mathfrak{A}_\Delta(\Gamma, \mathsf{G})$ has cocycle u then, Corollary 8.6 implies that for all $\gamma \in \Gamma$

$$\mathrm{d}\lambda^\gamma(\mathsf{u}) \in V_{\mathsf{H}} \cap \mathfrak{a}. \tag{15.1}$$

If we assume that $V_{\mathsf{H}} \cap \mathfrak{a} \subset \ker\psi$ then $\mathbb{V}^\psi_{\mathsf{u}} \subset \ker\psi$ and thus Remark 2.33 shows degeneracy.

If $V_{\mathsf{H}} \cap \mathfrak{a} \cap \ker\psi = \{0\}$ then necessarily $\dim V_{\mathsf{H}} \cap \mathfrak{a} = 1$ and $\mathbb{V}^\psi_{\mathsf{u}}$ is an interval contained in this line (possibly reduced to a point).

Let us deal now with item (ii), we first establish the result for ψ and deal afterwards with the general case. Since $\mathrm{d}\hbar^\psi(\mathsf{u}) = 0$ Eq. (2.19) gives that $\psi(p_\psi\mathsf{u}) = 0$, however $p_\psi\mathsf{u} \in \mathbb{V}^\psi_{\mathsf{u}} \subset V_{\mathsf{H}} \cap \mathfrak{a}$ which only intersects $\ker\psi$ at $\{0\}$. We conclude that

$$p_\psi\mathsf{u} = 0.$$

If $\mathbb{V}^\psi_{\mathsf{u}}$ were reduced to a point, we would obtain that for all $\gamma \in \Gamma$ $\mathrm{d}\lambda^\gamma(\mathsf{u}) = 0$ contrary to our assumptions. We obtain thus that $\mathbb{V}^\psi_{\mathsf{u}}$ is an interval with non-empty interior. This implies in turn that $\psi(\vec{\mathcal{J}})$ and $\psi(\mathcal{J})$ are not Livšic-cohomologous and thus that $0 = p_\psi\mathsf{u} \in \operatorname{int}\mathbb{V}^\psi_{\mathsf{u}}$. This gives item (ii) for ψ but also gives a bit more: there exists $\gamma, h \in \Gamma$ such that

$$\psi(\mathrm{d}\lambda^\gamma(\mathsf{u})) < 0 < \psi(\mathrm{d}\lambda^h(\mathsf{u})). \tag{15.2}$$

If we let now $\varphi \in \operatorname{int}(\mathcal{L}_\rho)^*$ be such that $V_{\mathsf{H}} \cap \mathfrak{a} \cap \ker\varphi = \{0\}$, then there exists $c \neq 0$ such that $\varphi|V_{\mathsf{H}} \cap \mathfrak{a} = c\psi|V_{\mathsf{H}} \cap \mathfrak{a}$. Assume without loss of generality that $c > 0$. Whence Equation (15.2) gives that

$$\varphi(\mathrm{d}\lambda^\gamma(\mathsf{u})) < 0 < \varphi(\mathrm{d}\lambda^h(\mathsf{u})),$$

which implies that $\varphi(\mathbb{V}^\varphi_{\mathsf{u}})$ is an interval that contains 0 in its interior, giving in turn the desired result.

We now deal with item (iii). If $\mathbf{P}^\psi(\mathsf{u}) = 0$ then Remark 2.33 implies that $\mathbb{V}^\psi_{\mathsf{u}}$ is contained in a level set of ψ, and is thus a point. If it where zero, for every $\gamma \in \Gamma$ we have $\mathrm{d}\lambda^\gamma(\mathsf{u}) = 0$. Since we assumed this was not the case, $\mathbb{V}^\phi_{\mathsf{u}} \neq \{0\}$.

We now focus on item (iv), se we assume H has rank 1. As before $V_{\mathsf{H}} \cap \mathfrak{a}$ is one-dimensional. Then one has:

- there exists $c > 0$ such that for every $u \in \mathfrak{a}_{\mathsf{H}}$ one has $\varphi(u) = c\psi(u)$,
- there exists $C \neq 0$ such that for every $v \in V_{\mathsf{H}} \cap \mathfrak{a}$ one has $\varphi(v) = C\psi(v)$.

Consequently, for every $\mathsf{v} \in H^1_{\mathrm{Ad}\,\rho}(\Gamma, V_{\mathsf{H}})$ Equation (15.1) implies that $\varphi(\vec{\mathcal{J}}_{\mathsf{v}}) = C\psi(\vec{\mathcal{J}}_{\mathsf{v}})$ and $\varphi(\mathcal{J}_\rho) = c\psi(\mathcal{J}_\rho)$. Moreover, Corollary 2.20 implies then that

$$\frac{\partial}{\partial t}\Big|_{t=0} \hbar^{\varphi}_{\rho_t}\varphi(\mathcal{J}_{\rho_t}) = \frac{C}{c}\frac{\partial}{\partial t}\Big|_{t=0} \hbar^{\psi}_{\rho_t}\psi(\mathcal{J}_{\rho_t}).$$

Thus,

$$\begin{aligned}
\mathbf{P}^{\varphi}_{\rho}(\mathsf{v}) &= \frac{\operatorname{var}\left(\frac{\partial}{\partial t}\Big|_{t=0} \hbar^{\varphi}_{\rho_t}\varphi(\mathcal{J}_{\rho_t}), m_{-\hbar^{\varphi}_{\rho}\varphi(\mathcal{J})}\right)}{\hbar^{\varphi}_{\rho}\int \varphi(\mathcal{J}_\rho) dm_{-\hbar^{\varphi}_{\rho}\varphi(\mathcal{J})}} \\
&= \frac{\operatorname{var}\left(\frac{C}{c}\frac{\partial}{\partial t}\Big|_{t=0} \hbar^{\psi}_{\rho_t}\psi(\mathcal{J}_{\rho_t}), m_{-\hbar^{\psi}_{\rho}\psi(\mathcal{J})}\right)}{\hbar^{\psi}_{\rho}\int \psi(\mathcal{J}_\rho) dm_{-\hbar^{\psi}_{\rho}\psi(\mathcal{J})}} \\
&= \frac{C^2}{c^2}\mathbf{P}^{\psi}_{\rho}(\mathsf{v}).
\end{aligned}$$

□

15.3. **Pressure forms along the Fuchsian locus I.** Consider a Fuchsian representation $\delta : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$, we have

$$\mathsf{T}_\rho \mathcal{H}_{\mathsf{A}_{d-1}}(S) = \bigoplus_{e \text{ exponent}} \mathsf{T}^e_{\delta}. \tag{15.3}$$

Let $e \in [\![1, d-1]\!]$ and let $q \in H^0(K^{e+1})$ be a holomorphic differential of degree $e+1$ on the Riemann surface associated to δ. The Hitchin parametrization provides a *normalized deformation*

$$\Psi(q) \in \mathsf{T}_\delta \mathcal{H}_{\mathsf{A}_{d-1}}(S),$$

as in Labourie-Wentworth [52], then [52, Corollary 3.5.2] implies that if we let $[\mathsf{u}]_{\Psi(q)} \in H^1_{\mathrm{Ad}\,\delta}\big(\pi_1 S, \mathfrak{sl}_d(\mathbb{R})\big)$ be the associated cocycle

$$[\mathsf{u}]_{\Psi(q)} \in \mathsf{T}^e_\delta. \tag{15.4}$$

Moreover, [52, Proposition 6.5.7] states that if p is a holomorphic differential of degree $f + 1 \neq e + 1$ then

$$\mathbf{P}^{\varpi_1}_{\delta}\big(\Psi(q), \Psi(p)\big) = 0. \tag{15.5}$$

We can now establish the following.

Lemma 15.5. *Let $\mathfrak{g}$ have type A, B, C, D, or G_2. Let $\delta \in \mathcal{H}_{\mathfrak{g}}(S)$ be a Fuchsian representation, then for every $\psi \in \operatorname{int}(\mathcal{L}_\delta)^*$ and exponents $e \neq f$ the subspaces T^e_δ and T^f_δ are $\mathbf{P}^{\psi}_{\delta}$-orthogonal.*

Proof. Since $\varkappa^e$ is 1-dimensional and does not lie in $\ker \varpi_1$ (Remark 14.9), there exist $c_e \in \mathbb{R} - \{0\}$ such that $\psi|\varkappa^e = c_e \varpi_1|\varkappa^e$.

Consider now $\mathsf{u} \in \mathsf{T}^e_\delta$ and $\mathsf{v} \in \mathsf{T}^f_\delta$, and write $[\mathsf{u}] = [\mathsf{u}]_{\Psi(q)}$ for some holomorphic differential q and similarly for v and a differential p. Corollary 8.6 implies that $\vec{\mathcal{J}}_{\mathsf{u}}$

has values in $\varkappa^e$ and $\vec{\partial}_{\mathsf{v}}$ has values in $\varkappa^f$, and thus

$$\psi(\vec{\partial}_{\mathsf{u}}) = c_e \varpi_1(\vec{\partial}_{\mathsf{u}}),$$
$$\psi(\vec{\partial}_{\mathsf{v}}) = c_f \varpi_1(\vec{\partial}_{\mathsf{v}}).$$

Moreover, since δ is Fuchsian the argument of (iv) of Lemma 15.4 yields

$$\mathbf{P}^{\psi}_{\delta}\big(\Psi(p), \Psi(q)\big) = \frac{c_e c_f}{c_1^2}\mathbf{P}^{\varpi_1}_{\delta}\big(\Psi(p), \Psi(q)\big) = 0,$$

by Equation (15.5).

This deals with types A, B, C and G_2, and for all exponents for the type D_k except $\kappa^{k-1,\mathrm{a}}$. However by § 14.3

$$H^1_{\mathrm{Ad}\,\delta}(\pi_1 S, \mathfrak{so}(k-1,k)) \oplus H^1_{\mathrm{Ad}\,\delta}(\pi_1 S, V_{k-1,\mathrm{a}})$$

is a decomposition consisting on fixed and anti-fixed point of the involution $\mathrm{d}_\delta \underline{i}^{\mathfrak{X}}$, which is an isometry of $\mathbf{P}^{\psi}$ by Lemma 2.32, yielding the result. □

15.4. **Proof of Theorem 15.1.** By means of Theorem 13.5, we proceed with an analysis according to the Zariski closure of $\rho(\pi_1 S)$. If $\rho(\pi_1 S)$ is Zariski-dense then Corollary 12.12 implies that every $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ induces a Riemannian $\mathbf{P}^{\psi}$.

At the other end, if $\rho = \delta$ is Fuchsian then we have

$$\mathsf{T}_\delta \mathcal{H}_{\mathfrak{g}}(S) = \bigoplus_{e \text{ exponent}} \mathsf{T}^e_\delta. \tag{15.6}$$

If follows from Lemma 15.5 that, for type A, B, C, D and G_2, the above decomposition is orthogonal for every pressure form $\mathbf{P}^{\psi}_{\delta}$. So we only need to understand each factor T^e_δ.

Item (i) from Lemma 15.4 implies that $\mathbf{P}^{\psi}$ degenerates on every T^e_δ with $\varkappa^e \subset \ker\psi$. We have to show thus non-degeneracy of $\mathbf{P}^{\psi}$ on the adjoint factors with $\psi(\kappa^e) \neq 0$. Let e be such that this happens.

Lemma 15.2 states that the set of normalized variations $\mathbb{V}^{\psi}_v$ has non-empty interior. If $\mathbf{P}^{\psi}$ degenerates in T^e_δ thenLemma 15.4(ii) states that $\mathbb{V}^{\psi}_v$ is reduced to a point, yielding thus a contradiction. This concludes the Fuchsian points.

We deal now with type A and intermediate strata, i.e. $\rho(\pi_1 S)$ has Zariski-closure either $\mathsf{SO}(n, n+1)$ or $\mathsf{PSp}(2n)$ according to the parity of d (we deal later with the G_2-case), let $\mathfrak{g}$ be the associated Lie algebra. In this situation, Proposition 14.5 states that there are only two adjoint factors that coincide with $\mathfrak{g} = \bigoplus_{e \text{ odd}} V_e$ and $V_{\mathfrak{g}} = \bigoplus_{e \text{ even}} V_e$.

The first factor is settled by Corollary 12.12, so we focus on the latter. Lemma 15.3 states that for any $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ the set $\mathbb{V}^{\psi}_v$ has non-empty interior. So the only possibility for $\mathbb{V}^{\psi}_v$ to be contained on a level set if ψ is that $V_{\mathfrak{g}} \subset \ker\psi$, or equivalently ψ is i-invariant. Thus if ψ is not i-invariant then both restrictions $\mathbf{P}^{\psi}|H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, \mathfrak{g})$ and $\mathbf{P}^{\psi}|H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_{\mathfrak{g}})$ give definite pressure forms. To conclude it suffices to show that

$$H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, \mathfrak{g}) \perp_{\mathbf{P}^{\psi}} H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_{\mathfrak{g}}).$$

However, by Corollary 2.32 the involution $\underline{i}^{\mathfrak{X}}$ is an isometry of $\mathbf{P}^{\psi}$ and the decomposition above coincides with the decomposition

$$\mathsf{T}_\rho \mathcal{H}_{\mathsf{A}_{d-1}}(S) = \mathrm{Fix}\, d_\rho \underline{i}^{\mathfrak{X}} \oplus \mathrm{Fix}(-d_\rho \underline{i}^{\mathfrak{X}}),$$

thus the decomposition is $\mathbf{P}^{\psi}$-orthogonal, giving non-degeneracy on $\mathsf{T}_\rho \mathcal{H}_{\mathsf{A}_{d-1}}(S)$.

We finally deal with the case where $\rho(\pi_1 S)$ has Zariski-closure $\phi_{\varpi_\alpha}(\mathsf{G}_2)$, this is the most involved case. In this situation by Proposition 14.5 there are three adjoint factors

$$\begin{aligned}\mathfrak{sl}_7(\mathbb{R}) &= V_3 \oplus \big(V_1 \oplus V_5\big) \oplus \big(V_2 \oplus V_4 \oplus V_6\big)\\ &= V_3 \oplus \phi_{\varpi_\alpha}(\mathfrak{G}_2) \oplus \big(V_2 \oplus V_4 \oplus V_6\big).\end{aligned}$$

By Corollary 12.12 $\mathbf{P}^\psi$ is non-degenerate on deformations along $V_1 \oplus V_5 = \phi_{\varpi_\alpha}(\mathfrak{G}_2)$. The factor $V_2 \oplus V_4 \oplus V_6$ is dealt with by means of item (i) of Lemma 15.3: it asserts that if u is a non-trivial cocycle with values on this factor then $\mathbb{V}_v^\psi \subset \varkappa^2 \oplus \varkappa^4 \oplus \varkappa^6$ has non-empty interior, thus, unless $\mathbb{V}_v^\psi \subset \ker\psi$, which corresponds to $\psi \circ \mathrm{i} = \psi$, the pressure form $\mathbf{P}^\psi$ is definite on the H^1. The factor V_3 is dealt with in item (ii) of Lemma 15.3.

This deals with the adjoint factors individually. As in the previous paragraph, the H^1 associated to even exponents and the H^1 associated to odd exponents are $\mathbf{P}^\psi$-orthogonal, so to prove non-degeneracy it remains to understand the restriction of $\mathbf{P}^\psi$ to $H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3) \oplus H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_1 \oplus V_5)$.

As each factor has already been dealt with, we consider non-vanishing u_3 and $\mathsf{u}_{1,5}$ in $H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3)$ and $H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_1 \oplus V_5)$ respectively and study the deformation associated to

$$\mathsf{u} = \mathsf{u}_3 + \mathsf{u}_{1,5} \in H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_3) \oplus H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_1 \oplus V_5).$$

We have to show that any $\psi \in \mathrm{int}\,(\mathcal{L}_\rho)^*$ verifies $\mathbf{P}^\psi(\mathsf{u}) \neq 0$.

Since we are studying odd exponents, we can restrict ourselves to $\mathsf{SO}(3,4)$ with Cartan subspace $\mathfrak{a}_{\mathfrak{so}(3,4)} = \mathbb{R}^3$ with Weyl chamber $\{a \in \mathbb{R}^3 : a_1 \geq a_2 \geq a_3 \geq 0\}$ and simple roots

$$\sigma_1(a) = a_1 - a_2,\ \ \sigma_2(a) = a_2 - a_3 \text{ and } \varepsilon_3(a) = a_3.$$

The subalgebra $\phi_{\varpi_\alpha}(\mathfrak{G}_2)$ has Cartan subspace $\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)} \subset \mathfrak{a}_{\mathfrak{so}(3,4)}$ given by

$$\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathfrak{G}_2)} = \{(a_1, a_2, a_1 - a_2) : a_1, a_2 \in \mathbb{R}\}$$

and simple roots $\{\sigma_1, \sigma_2\}$, see Figure 4. The proof of non-degeneracy is split into two cases:

(i) $\sigma_2 \in \operatorname{supp}\psi$,
(ii) $\sigma_2 \notin \operatorname{supp}\psi$.

We deal first with item (i), the proof uses Lemma 10.9 applied to $\psi = \varphi$, so we assume by contradiction that $\mathbf{P}^\psi(\mathsf{u}) = 0$, or equivalently that $\psi(\vec{\mathcal{J}})$ and $c\psi(\mathcal{J})$ are Livšic-cohomologous for some $c \neq 0$.

By Theorem 13.2 $\hbar_\rho^{\sigma_1} = \hbar_\rho^{\sigma_2} = 1$ so Theorem 2.19 implies that there exists $\gamma \in \pi_1 S$ such that

$$\sigma_2(\rho\gamma) < \sigma_1(\rho\gamma) = \varepsilon_3(\rho\gamma).$$

Since by assumption $\sigma_2 \in \operatorname{supp}\psi$ we have σ_2 strictly minimizes $\rho(\gamma)$ among $\operatorname{supp}\psi$ and we can apply Lemma 10.9 (with $\alpha = \sigma_2$). Moreover, by Proposition 13.4, every pair $g, h \in \pi_1 S$ with disjoint fixed points on $\partial\pi_1 S$ has images $\rho(g), \rho(h)$ that are Δ-transversally proximal. By means of Benoist [3, Proposition 5.1] , and because $\mathrm{i} = \mathrm{id}$ in this case, we find a subgroup $\Gamma' < \pi_1 S$ such that $\rho(\Gamma') \subset \phi_{\varpi_\alpha}(\mathsf{G}_2)$ is Zariski-dense, Δ-Anosov and has limit cone contained in $\{a : \sigma_2(a) < \sigma_1(a)\}$. So applying Lemma 10.9 we see that

$$\mathscr{V}\mathscr{J}_{v|\Gamma'} \subset \ker \sigma_2. \tag{15.7}$$

However, the representation $\mathrm{Ad}_{\mathsf{SO}(3,4)}|\phi_{\varpi_\alpha}(\mathsf{G}_2)$ is disjoined, indeed it has only two factors and one of them has more restricted weights than the other (recall Definition 6.3), whence Corollary 8.6 states that $\mathscr{V}\mathscr{J}_{v|\Gamma'}$ has non-empty interior and cannot be contained in $\ker \sigma_2$.

We now turn to item (ii), i.e. $\sigma_2 \notin \operatorname{supp}\psi$. Since we're working in $\mathsf{SO}(3,4)$ we think of ψ as an element of $\mathfrak{a}^*_{\mathsf{SO}(3,4)}$, which is spanned by the fundamental weights

$$\begin{aligned}\varpi_{\sigma_1}(a) &= a_1,\\ \varpi_{\sigma_2}(a) &= a_1 + a_2,\\ \varpi_{\varepsilon_3}(a) &= a_1 + a_2 + a_3.\end{aligned}$$

Since, by assumption $\sigma_2 \notin \operatorname{supp}\psi$, up to scaling ψ, which does not change the pressure form $\mathbf{P}^\psi$, we have that for some $b \in \mathbb{R}$, either of the following hold:

$$\begin{aligned}\psi &= \varpi_{\sigma_1} + b\varpi_{\varepsilon_3}\\ \psi &= b\varpi_{\sigma_1} + \varpi_{\varepsilon_3}.\end{aligned}$$

Assume the first one holds (the other is analogous), so $\psi = \varpi_{\sigma_1} + b\varpi_{\varepsilon_3}$.

The form ψ, on $\mathfrak{a}_{\phi_{\varpi_\alpha}(\mathsf{G}_2)} = \varkappa^1 \oplus \varkappa^5$, verifies

$$\psi(a_1, a_2, a_1 - a_2) = a_1 + b(a_1 + a_2 + (a_1 - a_2)) = (1 + 2b)\varpi_{\sigma_1}(a),$$

and one has moreover $\varpi_{\sigma_1}|\varkappa^3 = -\varpi_{\varepsilon_3}|\varkappa^3$ so, upon writing $v = v_3 + v_{1,5}$ in the decomposition $\varkappa^3 \oplus (\varkappa^1 \oplus \varkappa^5)$,

$$\begin{aligned}\psi(v) &= \psi(v_3) + \psi(v_{1,5})\\ &= -b\varpi_{\sigma_1}(v_3) + (1 + 2b)\varpi_{\sigma_1}(v_{1,5})\\ &= \varpi_{\sigma_1}\big(- bv_3 + (1 + 2b)v_{1,5}\big).\end{aligned}$$

Assuming by contradiction that there exists $c \neq 0$ such that for all $\gamma \in \pi_1 S$ one has $\psi(\mathrm{d}\lambda^\gamma(\mathsf{u})) = c\psi(\lambda(\rho\gamma))$ we obtain that, for all γ one has

$$\varpi_{\sigma_1}\Big(- b\mathrm{d}\lambda^\gamma(\mathsf{u})_3 + (1 + 2b)\mathrm{d}\lambda^\gamma(\mathsf{u})_{1,5}\Big) = c(1 + 2b)\varpi_{\sigma_1}\big(\lambda(\rho\gamma)\big).$$

Considering the cocycle $\mathsf{u}' = -b\mathsf{u}_3 + (1+2b)\mathsf{u}_{1,5}$ we get, by linearity of the Margulis invariant, that

$$\mathrm{d}\lambda^\gamma(\mathsf{u}') = -b\mathrm{d}\lambda^\gamma(\mathsf{u})_3 + (1 + 2b)\mathrm{d}\lambda^\gamma(\mathsf{u})_{1,5},$$

so one has

$$\varpi_{\sigma_1}(\mathrm{d}\lambda^\gamma(\mathsf{u}')) = c(1 + 2b)\varpi_{\sigma_1}\big(\lambda(\rho\gamma)\big).$$

Since $\rho(\pi_1 S)$ acts irreducibly on $\mathbb{R}^7$ we can apply Theorem 2.37 to obtain that the cocycle u' is trivial, giving in turn that either u_3 or $\mathsf{u}_{1,5}$ is trivial, contradicting our starting assumption. This completes the proof for types A, and thanks to the inclusions (13.1) we have also dealt with types B, C and G_2.

We end this section dealing with type D, so let $\rho \in \mathcal{H}(S, \mathsf{D}_n)$ with $n \geq 5$ (we deal later with D_4) and $\psi \in \operatorname{int}(\mathcal{L}_\rho)^*$.

As before we make use of the classification of Zariski closures of $\rho(\pi_1 S)$ given by Theorem 13.5 with the addition of Lemma 7.6 from Carvajales-Dey-Pozzetti-Wienhard [19]: Theorem 13.5 classifies the semi-simple part of the Zariski closures, [19, Lemma 7.6] states that the Zariski-closure is semi-simple. Thus we have that $\overline{\rho(\pi_1 S)}^{\mathrm{Z}}$ is:

- $\mathsf{SO}(n, n)$, in which case Corollary 12.12 implies that $\mathbf{P}^\psi$ is Riemannian;
- a principal SL_2, this case is dealt with by Lemmas 15.2 and 15.5;

- a representation $\mathsf{SO}(n-1,n) \to \mathsf{SO}(n,n)$ that stabilizes a non-isotropic line.

It remains to deal with the last item. In this case we have a group involution $\underline{i}$ of $\mathsf{SO}(n,n)$ whose fixed points are the corresponding $\mathsf{SO}(n-1,n)$. Thus, $\operatorname{Ad}\rho$ has two adjoint factors given by Equation (14.9), which are the fixed points and anti-fixed point set $d_e\underline{i}$; Corollary 2.32 implies that

$$H^1_{\operatorname{Ad}\rho}(\pi_1 S, \operatorname{Fix} d_e\underline{i}) \perp_{\mathbf{P}^\psi} H^1_{\operatorname{Ad}\rho}(\pi_1 S, \operatorname{AntiFix} d_e\underline{i}) \tag{15.8}$$

so we only have to deal with each factor independently.

One factor corresponds to deformations inside $\mathsf{SO}(n-1,n)$ and is settled by Corollary 12.12, the other one has one-dimensional neutralizing space, namely $\varkappa^{n-1,\mathrm{a}}$ which is dealt with by means of the combination if items (ii) and (iii) of Lemma 15.4. Indeed, we only need to find a linear form $\varphi \in \operatorname{int}(\mathcal{L}_\rho)^*$ whose entropy has vanishing derivative along a given $\mathsf{u} \in H^1_{\operatorname{Ad}\rho}(\pi_1 S, V_{n-1,\mathrm{a}})$. If we let $\mathfrak{a}_{\mathfrak{so}_{n,n}} = \mathbb{R}^n$ be a Cartan subspace of $\mathfrak{so}_{n,n}$ and consider the set of simple roots $\Delta = \{\sigma_1, \dots, \sigma_{n-1}, \alpha_n\}$ then the Cartan subspace of $\mathfrak{a}_{\mathfrak{so}_{n-1,n}}$ is $\ker(\sigma_{n-1} - \alpha_n)$. Since by Theorem 13.2 $\Delta \subset \mathcal{Q}_\eta$ for every $\eta \in \mathcal{H}_{\mathsf{D}_n}(S)$, the linear form $(1/2)(\sigma_{n-1} + \alpha_n)$ has critical entropy at ρ (the argument is verbatim from the G_2-case in Figure 6), as desired.

We finally deal with D_4. The above discussion holds verbatim, except that we have one more possibility of the Zariski closure of $\rho(\pi_1 S)$, namely the fundamental representation for the short root of $\mathsf{SO}(3,4)$. From § 14.3.1 the adjoint factors of $\operatorname{Ad}\rho$ are deformations along $\mathsf{SO}(3,4)$ and cocycles with values in $\underline{\tau}(V^{3,\mathrm{a}})$, these spaces correspond also to fixed and anti-fixed points of a Lie-algebra involution (namely $\underline{\tau i \tau^{-1}}$), and the above discussion works verbatim, giving, by Equation (14.11), the resulting condition for ψ to have degenerate pressure form.

16. Pressure forms at the Fuchsian locus II

We restrict ourselves now to $\mathfrak{g} = \mathfrak{sl}(d,\mathbb{R})$ and the natural inclusions from Equation (13.1).

Recall from Labourie-Wentworth [52] that if δ is a Fuchsian representation and q is a holomorphic differential over S_δ, then there is a natural tangent vector $\Psi(q) \in \mathsf{T}_\delta \mathcal{H}_{\mathsf{A}_{d-1}}(S)$, called the *normalized* deformation. We will use our techniques and the main result from [52] to homogenize the pressure metrics on these vectors.

16.1. **Description of pressure metrics at the Fuchsian locus.** Let us fix as reference the functional $\varpi_1 \in \mathfrak{a}^*$ and consider $\psi \in \mathfrak{a}^*$ with $\psi(\kappa^1) > 0$. For each $e \in [\![1, d-1]\!]$ we let $c_e \in \mathbb{R}$ be defined by

$$\psi|\varkappa^e := c_e \varpi_1|\varkappa^e.$$

Equivalently, by Remark 14.9, $c_e := \psi(\kappa^e)/(e!(d-1)^{\underline{e}})$. The pressure form $\mathbf{P}^\psi$ is defined on the open set $\mathcal{U}_\psi$ (recall Eq. (2.16)) which, since $c_1 > 0$, contains the Fuchsian locus. We describe $\mathbf{P}^\psi$ at the Fuchsian points.

Proposition 16.1. *For every Fuchsian δ and $v \in \mathsf{T}^e_\delta$ one has* $\mathbf{P}^\psi_\delta = \left(\dfrac{c_e}{c_1}\right)^2 \mathbf{P}^{\varpi_1}_\delta$.

Proof. It is exactly as in the proof of Lemma 15.4(iv). □

16.2. **A pressure form compatible at the Fuchsian locus.** Consider a Fuchsian representation $\delta : \pi_1 S \to \mathsf{PSL}_d(\mathbb{R})$, $e \in [\![1, d-1]\!]$ and $q \in H^0(K^{e+1})$ a holomorphic differential of degree $e+1$ on the Riemann surface associated to δ. Then Hitchin's parametrization provides a *normalized deformation*

$$\Psi(q) \in \mathsf{T}_\delta \mathcal{H}_{\mathsf{A}_{d-1}}(S)$$

as in Labourie-Wentworth [52] and one has the following.

Theorem 16.2 (Labourie-Wentworth [52, Cor. 3.5.2 and Cor. 6.1.2])**.** *Let δ be a Fuchsian representation, q a holomorphic differential on S_δ of degree $e+1$ and $[\mathsf{u}]_{\Psi(q)} \in H^1_{\mathrm{Ad}\,\delta}(\pi_S, \mathfrak{sl}_d(\mathbb{R}))$ the cocycle associated to the normalized deformation $\Psi(q)$. Then $[\mathsf{u}]_{\Psi(q)}$ has values in V_e. Moreover*

$$\begin{aligned}\mathbf{P}^{\varpi_1}_\delta\big(\Psi(q)\big) &= \frac{(d-1)!^2}{2^e}\frac{(d+1)d(d-1)}{3\cdot 2}\frac{(2e+1)!}{(d+e)!(d-e-1)!}\frac{\int_S \|q\|^2 \mathrm{darea}_\delta}{\pi|\chi(S)|}\\ &= \frac{(d-1)^{\underline{e}}}{(d+e)^{\underline{e-1}}}\frac{(d-1)}{3}\frac{(2e+1)!}{2^e}\frac{\int_S \|q\|^2 \mathrm{darea}_\delta}{2\pi|\chi(S)|}.\end{aligned}$$

We consider then the following functional.

Definition 16.3. We let $\varphi \in \mathfrak{a}^*$ be defined by, for all $e \in [\![1, d-1]\!]$,

$$\varphi|\varkappa^e = \sqrt{\frac{(d+e)^{\underline{e-1}}}{(d-1)^{\underline{e}}}\frac{3\cdot 2^e}{(\dim V_e)!(d-1)}} \cdot \varpi_1|\varkappa^e.$$

By Remark 14.9 $\varpi_1(\kappa^e) \neq 0$, whence by definition $\varphi(\kappa^e) \neq 0$ and thus Theorem 15.1 entails that $\mathbf{P}^\varphi$ is Riemannian on $\mathcal{U}_\varphi$. We have:

Corollary 16.4. *Let δ be a Fuchsian representation and $q \in \bigoplus_{e=1}^{d-1} H^0(K^e)$ then*

$$\mathbf{P}^\varphi_\delta\big(\Psi(q)\big) = \frac{(d-1)^2}{2\pi|\chi(S)|}\int_{S_\delta} \|q\|^2 \mathrm{darea}_\delta$$

and φ is the only linear form so that this equation holds at the Fuchsian points. Thus, there exists $\lambda > 0$ so that the operator j defined by $\mathbf{P}^\varphi(u,v) = \omega(\mathrm{j}u, v)$ squares $-\lambda$ on the tangent space to $\mathcal{H}_{\mathsf{A}_{d-1}}(S)$ at the Fuchsian points $\mathcal{T}(S)$.

Proof. Theorem 16.2 states that $[\mathsf{u}]_{\Psi(q)}$ belongs to $H^1_{\mathrm{Ad}\,\rho}(\pi_1 S, V_e)$, so the result follows from Theorem 16.2 and Proposition 16.1. The last assertion now follows from [52, Lemma 5.1.1]. ☐

If we want to find the form $\varphi_\mathfrak{g}$ restricted to the Hitchin components of type B, C or G_2, then we keep the coefficients of Definition 16.3 for odd exponents and impose $\varphi_\mathfrak{g}(\varkappa^e) = 0$ for even exponents, for type G_2 we further impose $\varphi_{\Phi_{\varpi_\alpha}(\mathsf{G}_2)}(\varkappa^3) = 0$.

Remark 16.5. Eq. (1.2) contains the computation of φ for the rank 2 simple split algebras, we compute here φ for rank 3. These computations are straightforward consequence of the definition of φ and the formulæ for κ^e from Lemma 14.8:

$$20\varphi_{\mathfrak{sl}(4,\mathbb{R})}(a) = \Big(6+\frac{\sqrt{10}}{15}\Big)a_1 + \Big(4-\frac{\sqrt{10}}{15}-\frac{\sqrt{10}\sqrt{3}}{3}\Big)a_2 + \Big(2+\frac{2\sqrt{10}}{15}-\frac{\sqrt{10}\sqrt{3}}{3}\Big)a_3;$$
$$35\varphi_{\mathfrak{sp}(6,\mathbb{R})}(a) = \Big(5+\frac{211\sqrt{35}}{3780}\Big)a_1 + \Big(3-\frac{299\sqrt{35}}{3780}\Big)a_2 + \Big(1-\frac{79\sqrt{35}}{1890}\Big)a_3;$$
$$28\varphi_{\mathfrak{so}(3,4)}(a) = \Big(3+\frac{\sqrt{42}\sqrt{10}}{90}+\frac{\sqrt{42}}{3780}\Big)a_1 + \Big(2-\frac{\sqrt{42}\sqrt{10})}{90}-\frac{\sqrt{42}}{945}\Big)a_2 + \Big(1-\frac{\sqrt{42}\sqrt{10}}{90}+\frac{\sqrt{42}}{756}\Big)a_3.$$

17. Hausdorff dimension degenerations

We begin by recalling the relation between Pressure forms and the Hessian of Hausdorff dimension from Bridgeman-Pozzetti-S.-Wienhard [16]. Let G be a simple split real-algebraic Lie groupe $\mathsf{G}_{\mathbb{C}}$ the group of its $\mathbb{C}$-points and J the almost complex structure on the space of complex characters $\mathfrak{X}(\Gamma, \mathsf{G}_{\mathbb{C}})$ induced by the complex structure of $\mathsf{G}_{\mathbb{C}}$.

For a simple root σ of $\mathsf{G}_{\mathbb{C}}$ (and of G) we consider the map

$$\begin{aligned}\mathrm{Hff}_\sigma : \mathfrak{A}_{\{\sigma\}}(\Gamma, \mathsf{G}_{\mathbb{C}}) &\to \mathbb{R}_{>0}\\ \rho &\mapsto \mathrm{Hff}\big(\xi^\sigma(\partial\Gamma)\big).\end{aligned}$$

It follows from Pozzetti-S.-Wienhard [62] and Bridgeman-Canary-Labourie-S. [14] that Hff_σ is an analytic function on a neighborhood $\mathcal{V}$ of the Hitchin component inside the complex characters:

$$\mathcal{H}_{\mathfrak{g}}(S) \subset \mathcal{V} \subset \mathfrak{X}(\pi_1 S, \mathsf{G}_{\mathbb{C}}).$$

If $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ then

$$\mathsf{T}_\rho \mathfrak{X}(\pi_1 S, \mathsf{G}_{\mathbb{C}}) = \mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S) \oplus \mathsf{J}\big(\mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)\big).$$

The function Hff_σ is constant on $\mathcal{H}_{\mathfrak{g}}(S)$ so we study the complementary factor $\mathsf{J}\big(\mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)\big)$.

Theorem 17.1 ([16])**.** *Let $\mathfrak{g}$ be simple split and $\rho \in \mathcal{H}_{\mathfrak{g}}(S)$ then, for every $v \in \mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)$ and $\sigma \in \Delta$ one has* $\mathrm{Hess}_\rho \mathrm{Hff}_\sigma(\mathsf{J}v) = \mathbf{P}^\sigma_\rho(v)$.

Together with Theorem C one obtains:

Corollary 17.2.

(i) *If $0 \neq v \in \mathsf{T}_\rho \mathcal{H}_{\mathfrak{g}}(S)$ has Zariski-dense base-point then,*
$$\mathrm{Hess}_\rho \mathrm{Hff}_\sigma(\mathsf{J}v) > 0.$$
(ii) *Let $v \in \mathsf{T}_\delta \mathcal{H}_{\mathfrak{g}}(S)$ be tangent to a Fuchsian representation δ. If $v \in \mathsf{T}^e_\delta$ and $\varkappa^e \subset \ker\sigma$, then $\mathrm{Hess}_\delta \mathrm{Hff}_\sigma(\mathsf{J}v) = 0$. If $\mathfrak{g}$ has classical type then the converse is also true: if $\mathrm{Hess}_\delta \mathrm{Hff}_\sigma(\mathsf{J}v) = 0$ then $v \in \bigoplus_{e:\varkappa^e \subset \ker\sigma} \mathsf{T}^e_\delta$.*

We now apply Theorem 15.1 to understand degenerations for the lower strata. The theorem reduces the question to two exceptional situations (i.e. $\phi_{\varpi_\alpha}(\mathsf{G}_2) \subset \mathsf{SL}(7,\mathbb{R})$ and the spin representation of $\mathsf{SO}(3,4)$ inside $\mathsf{SO}(4,4)$) and the Fuchsian locus for types A, B, C, D and G_2. Together with Theorem 17.1 the question is reduced to understanding the triplets (d, e, j) such that

$$\sigma_j(\kappa^e) = 0. \tag{17.1}$$

Such Kostant line will be called a *simple-singular Kostant line*. In the following we try to understand this equation in some situations, however the general case is still to be understood.

17.1. **Elementary Families.**

Proposition 17.3. *The following families are simple-singular Kostant lines for* A_{d-1}*:*

(i) $d = 2n$, *even exponent and the middle simple root* σ_n,
(ii) *the second root* σ_2 *and, for every exponent* e, $d = 1 + e(e+1)/2$,
(iii) *the triplets (d, exponent, root) defined as, for every* $m \in \mathbb{N}$

$$(4m+3, 2m+1, 2m),$$

(iv) $e = 3$ *and the pairs* $(d, j) := \begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 7 \\ 2 \end{pmatrix}$, *for any integer* $k \geq 0$,
(v) *the 4th exponent* $e = 4$ *and the pairs* (d, j) *of the form* $(d, j) = \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 11 \\ 2 \end{pmatrix}$, *or* $(d, j) = \begin{pmatrix} 6 & -7 \\ 1 & -1 \end{pmatrix}^k \begin{pmatrix} 17 \\ 3 \end{pmatrix}$, *for any integer* $k \geq 0$.

Proof. Item (i) follows from i-invariance of $\varkappa^e$ for even e (Remark 14.2). The next two items follow by direct computation on the formula from Lemma 14.8. Let us do (iii).

Indeed, using Lemma 14.8 and replacing j by $2m$, e by $2m+1$ and d by $4m+3$ one has

$$\begin{aligned}
\frac{\sigma_{2m}(\kappa^{2m+1})}{(-1)^e(e+1)!} &= \sum_{t=1}^{2m+1} (-1)^t \binom{2m+1}{t}\binom{2m+1}{t-1}(2m-1)^{\underline{2m+1-t}}(2m+2)^{\underline{t-1}} \\
&= \sum_{t=2}^{2m+1} (-1)^t \tbinom{2m+1}{t}\tbinom{2m+1}{t-1}\frac{(2m-1)!}{(t-2)!}\,\frac{(2m+2)!}{(2m+3-t)!} \\
&= (2m-1)!(2m+2)\sum_{t=2}^{2m+1} (-1)^t \tbinom{2m+1}{t}\tbinom{2m+1}{t-1}\tbinom{2m+1}{t-2} \\
&= 0,
\end{aligned}$$

since by the change of variables $k = 2m+1-t$ in the above formula, the sum equals its negative[6].

The next two also follow from Lemma 14.8 however in these cases one is left to find the integer solutions of an integral quadratic form $q = c$. For the third exponent one has $q_3(d, j) = -d^2 + 5dj - 5j^2 = 1$ which is solved by computing the cyclic group $\mathsf{SO}(q, \mathbb{Z}) = \langle \begin{pmatrix} 4 & -5 \\ 1 & -1 \end{pmatrix} \rangle$. The last case is analogous. □

[6]We thank Germain Poullot for the above argument leading to the combinatorial identity $\sum_{t=2}^{2m} (-1)^t \tbinom{2m+1}{t}\tbinom{2m+1}{t-1}\tbinom{2m+1}{t-2} = m(2m+1)^2$.

$d = \text{rank}+1$	exponent	root
46	32	21
70	49	34
128	90	62
153	132	75
156	34	4
571	494	284

TABLE 4. Simple-singular principal lines of A_{d-1}, up to $d = 700$, that do not fall in the elementary families or in Corollary 1.4.

17.2. **Degenerations for the 3rd root.** We establish in this section Corollary 1.4. By Theorem 17.1 we must describe the vectors $v \in \mathsf{T}_\rho \mathcal{H}_{\mathsf{A}_{d-1}}(S)$ such that $\mathbf{P}^{\sigma_3}(v) = 0$. If we moreover assume $d > 6$ then, Theorem 15.1 implies that $\rho = \delta$ is Fuchsian and

$$v \in \bigoplus_{e:\sigma_3(\varkappa^e)=0} \mathsf{T}^e_\delta.$$

We thus investigate each factor T^e_δ:

Corollary 17.4. *An element $v \in \mathsf{T}^e_\delta$ is such that $\mathbf{P}^{\sigma_3}(v) = 0$ if and only if the pair (d, e) satisfies the Diophantine equation*

$$e^4 - 6de^2 + 2e^3 + 6d^2 - 6de + 11e^2 - 18d + 10e + 12 = 0, \tag{17.2}$$

together with the constrain $1 < e < d$.

Proof. By means of Lemma 14.8 we obtain that $\sigma_3(\kappa^e) = 0$ if and only if

$$\sum_{t=1}^{e} (-1)^t \binom{e}{t}\binom{e}{t-1} 2^{\underline{e-t}} (d-4)^{\underline{t-1}} = 0. \tag{17.3}$$

We begin by observing that $2^{\underline{e-t}} \neq 0$ if and only if $e - t \in \{0, 1, 2\}$, and that $(d-4)^{\underline{t-1}} \neq 0$ iff $t < d-2$. Moreover, since $t \leq e$ and κ^{d-1} is never singular by Eq. (14.4), we can restrict to $t \leq e \leq d-2$.

If we let $p(e, t)$ denote the general term in the sum (17.3), then we want to compute the alternated sum

$$p(e, e) - p(e, e-1) + p(e, e-2) = 0,$$

together with the constrain $e \leq d-2$. Explicit computation gives

$$p(e, e) = \left(e(d-4)^{\underline{e-3}}\right)(d-e-1)(d-e-2);$$
$$p(e, e-1) = \left(e(d-4)^{\underline{e-3}}\right)e(e-1)(d-e-1);$$
$$p(e, e-2) = \left(e(d-4)^{\underline{e-3}}\right)\frac{e(e-1)^2(e-2)}{6}.$$

So alternating the sum and removing the common factor gives Equation (17.2). □

In order to complete the proof of Corollary 1.4 we explicitly solve (17.2) over $\mathbb{Z}$.

Proposition 17.5. *The integer solutions (d, e) of Equation (17.2) are precisely given by Table 5.*

e	d	d
-9	58	17
-5	17	6
-3	6	3
-2	3	2
-1	2	1
0	2	1
1	3	2
2	6	3
4	17	6
8	58	17

TABLE 5. All integer solutions of Equation (17.2), each e has two possible d's which appear in the subsequent two columns.

We prove now Proposition 17.5, i.e. we explicitly solve Equation (17.2) over the integers. A first remark is that it is preserved by the involution $(d, e) \mapsto (d, -e-1)$, so we only need to find (and care about) its solutions for positive e.

Clearing the variable d gives $d = \frac{e^2+e+3}{2} \pm \frac{1}{6}\sqrt{3(e^4+2e^3-e^2-2e+3)}$ so we now focus on the Diophantine equation

$$f(x) := 3(x^4 + 2x^3 - x^2 - 2x + 3) = y^2, \tag{17.4}$$

which, by means of the rational solution $(-1, 3)$, can be transformed by a $\mathbb{Q}$-birational map to an elliptic equation:

Lemma 17.6. *Consider the elliptic curve over $\mathbb{Q}$ defined by*

$$E: \; y^2 = x^3 - 147x + 610. \tag{17.5}$$

Then rational solutions of Equation (17.4) are parametrized by $E(\mathbb{Q})$ via the rational maps X_1 and X_2 defined by

$$X_1(x,y) = \left(-\frac{-7x+35+y}{17+y-x}, 3\frac{-2x^3+y^2+9x^2+28y+169}{(17+y-x)^2}\right);$$
$$X_2(x,y) = \left(-\frac{7x-35+y}{-17+x+y}, 3\frac{-2x^3+y^2+9x^2-28y+169}{(-17+x+y)^2}\right).$$

Proof. This is standard given there exists a rational solution of (17.4), in this case $x = -1$, $y = 3$. We begin by replacing x by $x - 1$ which gives

$$3x^4 - 6x^3 - 3x^2 + 6x + 9 - y^2 = 0,$$

followed by replacing x by $1/x$ and y by $3y/x^2$, which gives, by considering the numerator

$$9x^4 + 6x^3 - 3x^2 - 6x + 3 - 9y^2 = 0 = x^4 + \frac{2}{3}x^3 - \frac{1}{3}x^2 - \frac{2}{3}x + \frac{1}{3}x - y^2.$$

We now replace x by $x - 2/(3 \cdot 4)$ and y by $x^2 + y - (1/6 \cdot 2)$ to obtain a quadratic polynomial on x, indeed the additive terms are chosen to make disappear the higher-degree terms on x. The solutions of the obtained quadratic polynomial on x are

$$x = \frac{-14 \pm \sqrt{-5832y^3 + 2646y + 610}}{18(6y+1)},$$

which are obtained by describing the rational solutions of the equation $\Delta = v^2$. This latter equation is

$$v^2 = -5832u^3 + 2646u + 610 = \big(-3^2 2u\big)^3 - 147\big(-3^2 2u\big) + 610,$$

so replacing $-3^2 2u$ by u we obtain the desired elliptic Equation (17.5), and the composition of the above local change of variables give the stated rational map(s). □

We now proceed via the $\mathfrak{Ellog}$ method and, more precisely, use Tzanakis [75]. The Mordel-Weil group of $E(\mathbb{Q})$ consists on one torsion point $(5,0)$ (of order 2), and the points

$$R_1 = (9,4) \text{ and } R_2 = (11,18)$$

form a basis of the free part. If (x,y) is a rational solution of Equation (17.4) then via Lemma 17.5 $X_i(x,y) \in E(\mathbb{Q})$ and thus can be written as

$$X_i(x,y) = m_0(5,0) + m_1 R_1 + m_2 R_2$$

for some integers m_i, (with $m_0 \in \{0,1\}$), additivity here is the group law of the Mordel-Weil group of $E(\mathbb{Q})$. The method consists on providing a 'reasonable' upper bound for $M = \max\{|m_1|, |m_2|\}$ under the assumption that (x,y) is a pair of integers, which reduces the problem to an explicit computation that can be carried out by, for example, Maple.

To find this upper bound we collect some relevant data about the curve E, most of the following computations are computer-assisted and required to work on Maple with 13 decimal digits:

- The solutions of $q(u) = u^3 - 147u + 610 = 0$ are

  $$e_3 = \frac{-5 - 3\sqrt{57}}{2}, e_2 = 5, e_1 = \frac{-5 + 3\sqrt{57}}{2},$$

  and the minimal real period of E is

  $$\omega = 2\int_{e_3}^{\infty} \frac{dt}{\sqrt{q(t)}} \asymp 0.9810124566....$$

  We will also need the point $x_0 = 6\sqrt{3} - 1 > e_1$, we let $\sigma = 1$ and we consider the point

  $$R_0 = \big(x_0, 6(3 - \sqrt{3})\big) \in E(\mathbb{Q}(\sqrt{3})).$$

- If we let $E_0(\mathbb{R})$ denote the unbounded component, which is the identity component of the Mordell-Weil group of $E(\mathbb{R})$, then for $p \in E_0(\mathbb{R})$ with coordinates $(u(p), v(p))$, the map $\phi : E_0(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ given by

  $$\phi(p) := \begin{cases} 0 \mod 1 & \text{if } p = O, \\ \dfrac{1}{\omega}\displaystyle\int_{u(p)}^{\infty} \frac{du}{\sqrt{q(u)}} \mod 1 & \text{if } v(p) \geq 0, \\ -\phi(-P) \mod 1 & \text{if } v(p) \leq 0, \end{cases}$$

  is a group homomorphism.

- The discriminant of q is 2659392 and $\Delta = 2^4 * 2659392 = 42550272$,

- The j-invariant is $j_E = 470596/57$ and so the Archimedean contribution to its height is

$$h_\infty(j) = \log|470596/57| \asymp 9.018703988....$$

By means of [69, Theorem 1.1] we have, for every $p \in E(\mathbb{Q})$, that

$$\hat{h}(p) - \frac{1}{2}h(x(p)) \leq \frac{h(\Delta) + h_\infty(j)}{12} + 1.07 = c_{11} \asymp 3.285408400... \tag{17.6}$$

- The fundamental periods are, when denoting by M the arithmetic-geometric mean,

$$\omega_1 = \frac{2\pi}{\mathrm{M}(\sqrt{e_1 - e_3}, \sqrt{e_1 - e_2})} = 2\omega \asymp 1.962095763...,$$
$$\omega_2 = \frac{2\pi}{\mathrm{M}(\sqrt{e_1 - e_3}, \sqrt{-1}\sqrt{e_2 - e_3})} \asymp 1.177161295... - 1.128478211...\sqrt{-1}.$$

Since ω_2/ω_1 does not belong to the Gauss fundamental domain of the modular surface, we consider

$$\tau = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)(-\omega_2/\omega_1) \asymp 0.1849446113... + 1.171782212...\sqrt{-1}$$

with modulus $|\tau| \asymp 1.186287512...$
- Observe that R_1 and R_2 lie on the unbounded component $E_0(\mathbb{R})$. One has

$$\omega\phi(R_1) \asymp 0.8918445254...$$
$$\omega\phi(R_2) \asymp 0.6925571056...$$
$$\omega\phi(R_0) \asymp 0.8235278325...$$

- Let h be the logarithmic height, defined for a rational p/q in lowest terms, defined by $h(p/q) = \log\max\{|p|, |q|\}$, and if $(p_i/q_i) \in \mathbb{Q}^n$ we let

$$h(p_1/q_1, \ldots, p_n/q_n) = \log\max\{q, q|p_i|/q_i : i \in [\![1, n]\!]\},$$

where $q = \operatorname{lcm}\{q_i : i \in [\![1, n]\!]\}$. We denote by

$$h_E := \max\{1, h(-147/4, 610/16), h(j_E)\}$$
$$= h(j_E) = \log 470596 \asymp 13.06175526....$$

- The method from Tzanakis [75] requires us to choose numbers $A_0, \ldots, A_3, \mathcal{E}$ such that

$$A_0 \geq \max\left\{h_E, \frac{3\pi\omega^2}{|\omega_1|^2\Im(\tau)}\right\} = h_E \asymp 13.06175526....;$$
$$A_{i+1} \geq \max\left\{h_E, \frac{3\pi\omega^2\phi(R_i)^2}{|\omega_1|^2\Im(\tau)}, \hat{h}(R_i) : \right\}$$
$$= \max\left\{h_E, \hat{h}(R_i)\right\} = h_E;$$
$$\mathsf{e} \leq \mathcal{E} \leq \mathsf{e}\min\left\{\frac{|\omega_1|}{\omega}\cdot\sqrt{\frac{2A_0\Im\tau}{3\pi}}, \frac{|\omega_1|}{\omega\phi(R_i)}\cdot\sqrt{\frac{2A_{i+1}\Im\tau}{3\pi}}, i = 0, 1, 2\right\};$$

where we have used Equation (17.6) to find an upper bound of $\hat{h}(R_i)$, and e is the Euler number. Explicit computation implies we can choose then

$A_i = 13.5$ for $i = 0, 1, 2, 3$, and $\mathcal{E} = 9$. Then we compute the constants c_4, c_5 and c_6 from [75, § 7] given by David [22, Théorème 2.1]:

$$\begin{aligned} c_4 &= 2.9 \cdot 10^{30} \cdot 2^{10} \cdot 4^{32} \cdot 5^{80.3} (\log \mathcal{E})^{-9} (13.5)^4 \\ &\asymp 2.043497279... \cdot 10^{110} \\ c_5 &= \log(2\mathcal{E}) = \log(18) \\ c_6 &= \log(18) + h_E \asymp 15.95212702.... \end{aligned}$$

- We also consider the regulator matrix associated to the basis $\{R_1, R_2\}$, defined by the matrix associated to the quadratic form on $E(\mathbb{Q})$ defined by $\langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q)$, and we let c_1 be its smallest eigenvalue, explicit computation gives
$$c_1 \le 0.303868...$$
- We now have to find a constant c_9 such that
$$\frac{1}{\omega}\int_U^\infty \frac{dx}{\sqrt{f(u)}} \le \frac{c_9}{\omega}\frac{1}{U}$$
which can be easily shown to be
$$c_9 = \frac{1}{\sqrt{3}}.$$
- We now compute a constant c_{10} so that for every $u \ge 1$
$$h\Big(\frac{6\sqrt{Q(u)} - 6u + 18}{u^2}\Big) \le c_{10} + 2\log u,$$
and we get $c_{10} = 3$.
- We finally have that the constants $c_{12} = 1$ and $c_{13} = 0$ and we obtain the first upper bound on $M \ge 16$:
$$c_1 M^2 \le \log c_9 + \frac{1}{2}c_{10} + c_{11} + c_4(\log M + c_5)(\log\log M + c_6)^5,$$
which gives $M \le 6.123 \cdot 10^{59}$.
- We now proceed with the reduction of the upper bound for M applying [75, § 5]. Since R_0 does not belong to $E(\mathbb{Q})$, the $\mathbb{R}/\mathbb{Z}$ elements
$$\{\phi(R_0), \phi(R_1), \phi(R_2)\}$$
are linearly independent over $\mathbb{Q}$, and thus our situation is Case 2 in that section, we are hence bound to use [75, Proposition 4], which gives the reduction $M \le 30$. At this point we proceed with a case by case computation using, for example, Maple.

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