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Anosov representations and dominated splittings

Abstract. We provide a link between Anosov representations introduced by Labourie and dominated splitting of linear cocycles. This allows us to obtain equivalent characterizations for Anosov representations and to recover recent results due to Guéritaud–Guichard–Kassel–Wienhard [GGKW] and Kapovich–Leeb–Porti [KLP$_2$] by different methods. We also give characterizations in terms of multicones and cone types inspired by the work of Avila–Bochi–Yoccoz [ABY] and Bochi–Gourmelon [BG]. Finally, we provide a new proof of the higher rank Morse Lemma of Kapovich–Leeb–Porti [KLP$_2$].

Contents

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1. Introduction

The aim of this paper is to point out and exploit connections between the following two areas:

1. linear representations of discrete groups;
2. differentiable dynamical systems.

More specifically, we show that *Anosov representations* are closely related to *dominated splittings*. This relation allows us to recover some results about Anosov representations, and to give new characterizations of them.

Anosov representations were introduced by Labourie [Lab] in his study of the Hitchin component [Hit]. They provide a stable class of discrete faithful representations of word-hyperbolic groups into semisimple Lie groups, which unifies examples of varying nature. Since then, Anosov representations have become a prominent object of study, featuring in various deep results (see for example [GW], the surveys [BCS, Wie] and references therein). Recently, new characterizations of Anosov representations have been found in [GGKW, KLP$_1$, KLP$_2$, KLP$_3$], considerably simplifying the definition. It is now fairly generally accepted that Anosov representations are a good generalization of convex co-compact groups to higher rank.

In differentiable dynamical systems, the notion of *hyperbolicity* as introduced by Anosov and Smale [Sma] plays a central role. Early on, it was noted that weaker forms of hyperbolicity (partial, nonuniform, etc.) should also be studied: see [BDV] for a detailed account. One of these is the notion of *dominated splitting*, which can be thought of as a projective version of hyperbolicity: see Section 2.

As mentioned above, in this paper we benefit from the viewpoint of differentiable dynamics in the study of linear representations. For example, it turns out that a linear representation is Anosov if and only if its associated linear flow has a dominated splitting: see Subsection 4.3 for the precise statements.

Let us summarize the content of this paper.

In Section 2 we describe the basic facts about dominated splittings that will be used in the rest of the paper. We present the characterization of dominated splittings given by [BG, Theorem A]. We rely on that theorem in different contexts throughout the paper.

In Section 3 we introduce *dominated representations* ρ of a given finitely generated group Γ into GL($d, \mathbb{R}$). The definition is simple: we require the gap between some consecutive singular values to be exponentially large with respect to the word-length of the group element, that is,

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} < Ce^{-\lambda|\gamma|} \quad \text{for all } \gamma \in \Gamma,$$

for some constants $p \in \{1, \ldots, d-1\}$ and $C, \lambda > 0$ independent of $\gamma \in \Gamma$.

We show that the existence of dominated representations implies that the group Γ is word-hyperbolic: see Theorem 3.2. Word-hyperbolicity allows us to consider Anosov representations, and in Section 4 we show that they are exactly the same as dominated representations. Many of the results in Sections 3 and 4 are not new, appearing in the recent

works of [GGKW] and [KLP$_1$, KLP$_2$, KLP$_3$] with different terminology. Our proofs are different and make use of the formalism of linear cocycles and most importantly of nontrivial properties of dominated splittings. At the end of Section 4 we pose a few questions where the connection with differentiable dynamics is manifest.

In Section 5 we give yet another equivalent condition for a representation to be Anosov, which uses the sofic subshift generated by the cone types instead of the geodesic flow of the group: see Theorem 5.9. This condition is very much inspired by [ABY, BG] and provides nice ways to understand the variation of the limit maps, as well as a quite direct method to check if a representation is dominated. This criterion is used in Section 6 to recover a basic result from [BCLS] on the analyticity of the limit maps for Anosov representations: see Theorem 6.1.

Section 7 shows how [BG, Theorem A] implies a Morse-Lemma-type statement for the symmetric space of PSL$(d, \mathbb{R})$. That result is contained in the recent work of [KLP$_2$], but we provide a different approach. This section only relies on Section 2.

In Section 8 we replace GL$(d, \mathbb{R})$ with a real-algebraic noncompact semisimple Lie group. Representation theory of such groups allows one to reduce most of the general statements to the corresponding ones in GL$(d, \mathbb{R})$ by a fairly standard procedure. Nevertheless, more work is needed to obtain a Morse Lemma for symmetric spaces of noncompact type; this occupies most of Section 8.

The sections are largely independent; dependence is indicated in Fig. 1.
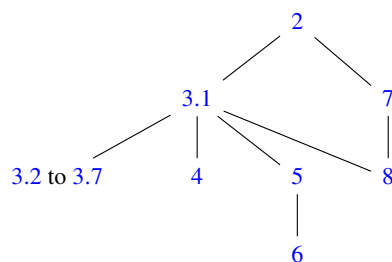
Fig. 1. Dependence between sections.

The results of this paper were announced at [EHY].

2. Dominated splittings

In the 1970s, Mañé introduced the notion of *dominated splitting*, which played an important role in his solution of Smale's Stability Conjecture: see [Sam] and references therein. Independently, dominated splittings had been studied in the theory of ordinary differential equations by the Russian school at least since the 1960s, where they were called *exponential separation*: see [Pa] and references therein. Dominated splittings continue to be an important subject in dynamical systems [CP, Sam] and control theory [CK].

2.1. Definition and basic properties of dominated splittings

Let X be a compact metric space. Let $\mathbb{T}$ be either $\mathbb{Z}$ or $\mathbb{R}$. Consider a continuous action of $\mathbb{T}$ on X, that is, a continuous family $\{\phi^t \colon X \to X\}_{t \in \mathbb{T}}$ of homeomorphisms such that $\phi^{t+s} = \phi^t \circ \phi^s$. We call $\{\phi^t\}$ a *continuous flow*.

Let E be a real vector bundle with projection map $\pi \colon E \to X$ and fibers $E_x :=$ $\pi^{-1}(x)$ of constant dimension d. We endow E with a Riemannian metric (that is, a continuous choice of an inner product on each fiber). Suppose $\{\psi^t \colon E \to E\}_{t \in \mathbb{T}}$ is a continuous action of $\mathbb{T}$ on E by automorphisms of the vector bundle that covers $\{\phi^t\}$, that is, $\pi \circ \psi^t = \phi^t \circ \pi$. So the restriction of ψ^t to each fiber E_x is a linear automorphism ψ^t_x onto $E_{\phi^t(x)}$. We say that $\{\psi^t\}$ is a *linear flow* which *fibers over* the continuous flow $\{\phi^t\}$.

The simplest situation is when $\mathbb{T} = \mathbb{Z}$ and the vector bundle is trivial, i.e., $E = X \times \mathbb{R}^d$ and $\pi(x, v) = x$; in that case the linear flow $\{\psi^t\}$ is called a *linear cocycle*, and in order to describe it, it is sufficient to specify the maps $\phi = \phi^1 \colon X \to X$ and $A \colon X \to \mathrm{GL}(d, \mathbb{R})$ such that $\psi^1(x, v) = (\phi(x), A(x)v)$. With some abuse of terminology, we sometimes call the pair (ϕ, A) a linear cocycle.

Suppose that the vector bundle E splits as a direct sum $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ of continuous[1] subbundles of constant dimensions.[2] This splitting is called *invariant* under the linear flow $\{\psi^t\}$ if for all $x \in X$ and $t \in \mathbb{T}$,

$$\psi^t(E^{\mathrm{cu}}_x) = E^{\mathrm{cu}}_{\phi^t(x)}, \qquad \psi^t(E^{\mathrm{cs}}_x) = E^{\mathrm{cs}}_{\phi^t(x)}.$$

Such a splitting is called *dominated* (with E^{cu} dominating E^{cs}) if there are constants $C, \lambda > 0$ such that for all $x \in X$, $t > 0$, and unit vectors $v \in E^{\mathrm{cs}}_x$, $w \in E^{\mathrm{cu}}_x$ we have

$$\frac{\|\psi^t(v)\|}{\|\psi^t(w)\|} < Ce^{-\lambda t}. \tag{2.1}$$

Note that this condition is independent of the choice of the Riemannian metric for the bundle. It is actually equivalent to the following condition (see [CK, p. 156]): for all $x \in X$ and all unit vectors $v \in E^{\mathrm{cs}}_x$, $w \in E^{\mathrm{cu}}_x$ we have

$$\lim_{t \to +\infty} \frac{\|\psi^t(v)\|}{\|\psi^t(w)\|} = 0. \tag{2.2}$$

The bundles of a dominated splitting are unique given their dimensions; more generally:

Proposition 2.1. *Suppose a linear flow* $\{\psi^t\}$ *has dominated splittings* $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ *and* $F^{\mathrm{cu}} \oplus F^{\mathrm{cs}}$, *with* E^{cu} *(resp.* F^{cu}*) dominating* E^{cs} *(resp.* F^{cs}*). If* $\dim E^{\mathrm{cu}} \leq \dim F^{\mathrm{cs}}$ *then* $E^{\mathrm{cu}} \subset F^{\mathrm{cs}}$ *and* $E^{\mathrm{cs}} \supset E^{\mathrm{cu}}$.

See e.g. [CP] for a proof of this and other properties of dominated splittings.

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[1] In fact, continuity of the bundles follows from condition (2.1): see e.g. [CP].
[2] cu and cs stand for *center-unstable* and *center-stable* respectively. This terminology is usual in differentiable dynamics.

One can also define domination for invariant splittings into more than two bundles. This leads to the concept of *finest dominated splitting*, whose uniqueness is basically a consequence of Proposition 2.1; see [BDP, CP].

The existence of a dominated splitting can be characterized in terms of cone fields (see e.g. [CP]); we will use these ideas later in Section 5.

2.2. Domination in terms of singular values

Another indirect way to detect the existence of a dominated splitting can be formulated in terms of the "nonconformality" of linear maps. Results of this kind were obtained in [Yo, Len] for dimension 2, and later in [BG] in more generality.[3] Let us explain this characterization.

If A is a linear map between two inner product vector spaces of dimension d, then its *singular values*

$$\sigma_1(A) \geq \cdots \geq \sigma_d(A)$$

are the eigenvalues of the positive semidefinite operator $\sqrt{A^*A}$, repeated according to multiplicity. They equal the semiaxes of the ellipsoid obtained as the A-image of the unit ball; this is easily seen by using the singular value decomposition [HJ, §7.3].

If $p \in \{1, \ldots, d-1\}$ and $\sigma_p(A) > \sigma_{p+1}(A)$, then we say that *$A$ has a gap of index p*. In that case, we denote by $U_p(A)$ the p-dimensional subspace containing the p largest axes of the ellipsoid $\{Av : \|v\| = 1\}$. Equivalently, $U_p(A)$ is the eigenspace of $\sqrt{AA^*}$ corresponding to the p largest eigenvalues. We also define $S_{d-p}(A) := U_{d-p}(A^{-1})$. Note that $S_{d-p}(A)^\perp = A^{-1}(U_p(A))$ and $U_p(A)^\perp = A(S_{d-p}(A))$ (see Fig. 2).
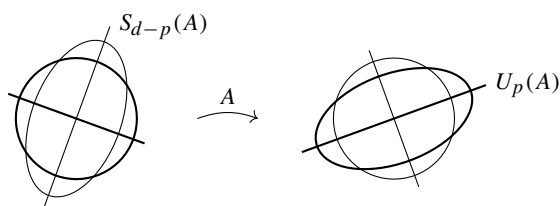
Fig. 2. Spaces associated to a linear map A.

The following theorem asserts that the existence of a dominated splitting can be detected in terms of singular values, and also describes the invariant subbundles in these terms:

Theorem 2.2 (Bochi–Gourmelon [BG]). *Let $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$. Let $\{\psi^t\}_{t \in \mathbb{T}}$ be a linear flow on a vector bundle E, fibering over a continuous flow $\{\phi^t\}_{t \in \mathbb{T}}$ on a compact metric space X. Then the linear flow $\{\psi^t\}$ has a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p if and only if there exist $c, \lambda > 0$ such that for every $x \in X$ and $t \geq 0$ we have*

$$\frac{\sigma_{p+1}(\psi_x^t)}{\sigma_p(\psi_x^t)} < ce^{-\lambda t}.$$

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[3] For a recent generalization to Banach spaces, see [BM].

Moreover, the bundles are given by

$$E_x^{\mathrm{cu}} = \lim_{t \to +\infty} U_p(\psi_{\phi^{-t}(x)}^t), \tag{2.3}$$

$$E_x^{\mathrm{cs}} = \lim_{t \to +\infty} S_{d-p}(\psi_x^t), \tag{2.4}$$

and these limits are uniform.

(To make sense of the limits above it is necessary to metrize the Grassmann bundle associated to E; the particular way of doing so is irrelevant to the statement.)

In [BG], the Multiplicative Ergodic Theorem of Oseledets is used to basically reduce the proof of Theorem 2.2 to some angle estimates. Since we will explicitly need such estimates in other parts of this paper, we will also present a sketch of the proof of Theorem 2.2 in §A.4.

Remark 2.3. Consider the case of *complex* vector spaces, bundles, etc. Dominated splittings can be defined analogously; so can singular values and the subspaces U_p, S_{d-p}. Theorem 2.2 also extends to the complex case; indeed, it can be deduced from the real case.

2.3. Domination for sequences of matrices

Next, we describe some consequences of Theorem 2.2 for sequences of $d \times d$ matrices.[4]

Given $K > 1$, define the following compact set:

$$\mathcal{D}(K) := \{A \in \mathrm{GL}(d, \mathbb{R}) : \|A\| \le K, \ \|A^{-1}\| \le K\}.$$

If I is a (possibly infinite) interval in $\mathbb{Z}$, we endow $\mathcal{D}(K)^I$ with the product topology, which is compact and is induced e.g. by the following metric:

$$d((A_n), (B_n)) := \sum_{n \in I} 2^{-|n|}(\|A_n - B_n\| + \|A_n^{-1} - B_n^{-1}\|).$$

Let $p \in \{1, \dots, d-1\}$ and $\mu, c > 0$. For each interval $I \subset \mathbb{Z}$, let $\mathcal{D}(K, p, \mu, c, I)$ denote the set of sequences $(A_n) \in \mathcal{D}(K)^I$ such that for all $m, n \in I$ with $m \ge n$ we have

$$\frac{\sigma_{p+1}}{\sigma_p}(A_m \cdots A_{n+1} A_n) \le c e^{-\mu(m-n+1)}.$$

Let $I = \mathbb{Z}$. Let ϑ denote the shift map on the space $\mathcal{D}(K, p, \mu, c, \mathbb{Z})$, and let $A \colon \mathcal{D}(K, p, \mu, c, \mathbb{Z}) \to \mathrm{GL}(d, \mathbb{R})$ denote the projection on the zeroth coordinate. The pair (ϑ, A) determines a linear cocycle (in the sense explained in §2.1). Note that the hypothesis of Theorem 2.2 is automatically satisfied. So we obtain:[5]

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[4] We note that similar sequences have been considered in [GGKW], namely sequences that have what they call *coarsely linear increments* (CLI) in certain Cartan projections.

[5] For a similar statement with different notation and not relying on Theorem 2.2, see [GGKW, Theorem 5.3].

Proposition 2.4. *Fix constants $K > 1$, $p \in \{1, \ldots, d-1\}$, and $\mu, c > 0$. Then, for each sequence $x = (A_n) \in \mathcal{D}(K, p, \mu, c, \mathbb{Z})$, the limits*

$$E^{cu}(x) := \lim_{n \to +\infty} U_p(A_{-1} A_{-2} \cdots A_{-n}),$$

$$E^{cs}(x) := \lim_{n \to +\infty} S_{d-p}(A_{n-1} A_{n-2} \cdots A_0)$$

exist and are uniform over $\mathcal{D}(K, p, \mu, c, \mathbb{Z})$. Moreover, $E^{cu} \oplus E^{cs}$ is a dominated splitting for the linear cocycle (ϑ, A) defined above.

By a compactness argument, the theorem above yields information for finite sequences of matrices (throughout, $\mathbb{N} = \{0, 1, 2, \ldots\}$):

Lemma 2.5. *Given $K > 1$ and $\mu, c > 0$, there exist $\ell_1 \in \mathbb{N}$ and $\delta > 0$ with the following properties. Suppose $I \subset \mathbb{Z}$ is an interval and $(A_i)_{i \in I}$ is an element of $\mathcal{D}(K, p, \mu, c, I)$. If $n < k < m$ all belong to I and $\min\{k - n, m - k\} > \ell_1$ then*

$$\angle\big(U_p(A_{k-1} \cdots A_{n+1} A_n), \ S_{d-p}(A_{m-1} \cdots A_{k+1} A_k)\big) > \delta.$$

Proof. The proof is by contradiction. Assume that there exist numbers $K > 1$ and $\mu, c > 0$ and sequences $\ell_j \to \infty$ and $\delta_j \to 0$ such that for each j there exist an interval $I_j \subset \mathbb{Z}$, an element $(A_i^{(j)})_{i \in I_j}$ of $\mathcal{D}(K, p, \mu, c, I_j)$, and integers $n_j < k_j < m_j$ in I_j such that $\min\{k_j - n_j, m_j - k_j\} > \ell_j$ and

$$\angle\big(U_p(A_{k_j-1}^{(j)} \cdots A_{n_j}^{(j)}), \ S_{d-p}(A_{m_j-1}^{(j)} \cdots A_{k_j}^{(j)})\big) \leq \delta_j.$$

Shifting indices, we can assume that $k_j = 0$ for every j. By a diagonal argument, passing to subsequences we can assume that for each $i \in \mathbb{Z}$, the matrices $A_i^{(j)}$ (which are defined for sufficiently large j) converge to some matrix A_i as $j \to \infty$. The resulting sequence $x = (A_i)_{i \in \mathbb{Z}}$ belongs to $\mathcal{D}(K, p, \mu, c, \mathbb{Z})$. If $E^{cu}(x)$ and $E^{cs}(x)$ are the limit spaces as in Proposition 2.4, then their angle is zero, which contradicts domination. $\square$

3. Domination implies word-hyperbolicity

In this section, we define dominated linear representations of a finitely generated group, and prove that groups that admit such representations are word-hyperbolic.

3.1. Dominated representations and the word-hyperbolicity theorem

Let Γ be a finitely generated group. Fix a symmetric generating set S of Γ. We denote by $|\gamma|$ the *word-length* of $\gamma \in \Gamma$, i.e. the minimum number of elements of S required to obtain γ as a product of elements of S. The *word-metric* is defined as

$$d(\gamma, \eta) := |\eta^{-1}\gamma|. \tag{3.1}$$

Then the action of Γ into itself by left multiplication is isometric.

Recall that the group Γ is called *word-hyperbolic* if it is a Gromov-hyperbolic metric space when endowed with the word-metric (3.1); this does not depend on the choice of the generating set S (see [Gr, CDP, GH, BH]).

A representation $\rho\colon \Gamma \to \mathrm{GL}(d,\mathbb{R})$ is *p-dominated* if there exist constants $C, \lambda > 0$ such that

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} \leq C e^{-\lambda|\gamma|} \quad \text{for all } \gamma \in \Gamma. \tag{3.2}$$

It is easy to see that the definition does not depend on the choice of S though the constants C and λ may change.

Remark 3.1. Since $\sigma_i(A) = \sigma_{d+1-i}(A^{-1})^{-1}$ for each i, the property $|\gamma^{-1}| = |\gamma|$ implies that if a representation ρ is p-dominated, then it is also $(d-p)$-dominated.

The purpose of this section is to show:

Theorem 3.2. *If a group Γ admits a p-dominated representation in $\mathrm{GL}(d,\mathbb{R})$ then Γ is word-hyperbolic.*

Theorem 3.2 follows from a more general result recently obtained by Kapovich, Leeb, and Porti [KLP$_2$, Theorem 1.4]. Their result concerns not only dominated representations, but quasi-isometric embeddings of metric spaces satisfying a condition related to domination. Here we use the results from Section 2 to give a direct and more elementary proof.

Let us mention that a related but different notion of domination was recently studied by other authors [DT, GKW].

Remark 3.3 (Representations in $\mathrm{SL}(d,\mathbb{R})$). Given a representation with target group $\mathrm{GL}(d,\mathbb{R})$, we can always assume that it has its image contained in matrices with determinant ± 1 by composing with the homomorphism $A \mapsto |\det A|^{-1/d} A$, which does not affect p-domination.

Remark 3.4 (Representations in $\mathrm{PGL}(d,\mathbb{R})$). We can define p-dominated representations in $\mathrm{PGL}(d,\mathbb{R})$ in exactly the same way, since the quotients σ_{p+1}/σ_p are well-defined in the latter group. Obviously, composing any p-dominated representation in $\mathrm{GL}(d,\mathbb{R})$ with the quotient map $\pi\colon \mathrm{GL}(d,\mathbb{R}) \to \mathrm{PGL}(d,\mathbb{R})$ we obtain a p-dominated representation. Conversely, given any p-dominated representation $\rho\colon \Gamma \to \mathrm{PGL}(d,\mathbb{R})$, we can find a group $\hat{\Gamma}$, a 2-to-1 homomorphism $f\colon \hat{\Gamma} \to \Gamma$, and a p-dominated representation $\hat{\rho}\colon \hat{\Gamma} \to \mathrm{GL}(d,\mathbb{R})$ (with determinants ± 1) such that $\pi \circ \hat{\rho} = f \circ \rho$. Theorem 3.2 shows that $\hat{\Gamma}$ is word-hyperbolic, and it follows (see [GH, p. 63]) that Γ is word-hyperbolic as well.

Remark 3.5 (General semisimple Lie groups). By using exterior powers, any p-dominated representation in $\mathrm{GL}(d,\mathbb{R})$ induces a 1-dominated representation in $\mathrm{GL}(k,\mathbb{R})$ for $k = \binom{d}{p}$. In Subsection 8.5 we shall see that every representation $\rho\colon \Gamma \to G$, where G is an arbitrary (real-algebraic noncompact) semisimple Lie group, can be reduced to the case of $\mathrm{PGL}(d,\mathbb{R})$ for some d via a similar construction.

Remark 3.6. Given Theorem 3.2, one may wonder whether every hyperbolic group admits a dominated representation. This is far from true, since there exist hyperbolic groups that are nonlinear, i.e. every linear representation of such a group factors through a finite group: see [Kap, Section 8]. On the other hand, one can ask if every linear hyperbolic group admits a dominated representation.

3.2. Criterion for hyperbolicity

In order to show word-hyperbolicity the following sufficient condition will be used:

Theorem 3.7 (Bowditch [Bow]). *Let Γ be a group which acts by homeomorphisms on a perfect compact metrizable topological space M in such a way that the diagonal action of Γ on the (nonempty) space*

$$M^{(3)} := \{(x_1, x_2, x_3) \in M^3 : x_i \neq x_j \text{ if } i \neq j\}$$

is properly discontinuous and cocompact. Then Γ is word-hyperbolic.

We recall that a continuous action of Γ on a topological space X is:

- *properly discontinuous* if given any compact subset $K \subset X$ there exists n such that if $|\gamma| > n$ then $\gamma K \cap K = \emptyset$;
- *cocompact* if there exists a compact subset $K \subset X$ such that $\Gamma x \cap K \neq \emptyset$ for every $x \in X$.

Remark 3.8. Theorem 3.7 also implies that the set M is equivariantly homeomorphic to $\partial \Gamma$. Here $\partial \Gamma$ denotes the *visual boundary* of the group Γ, defined as the set of equivalence classes of quasi-geodesic rays (i.e. quasi-geodesic maps from $\mathbb{N}$ to Γ) under the equivalence relation defined as being at finite Hausdorff distance from each other; see for example [GH, Chapitre 7] or [CDP, Chapitre 2]). The topology in $\partial \Gamma$ is given by pointwise convergence of the quasi-geodesic rays with the same constants and starting at the same point. In Section 6 we will also comment on the metric structure on the boundary.

A group is called *elementary* if it is finite or virtually cyclic; elementary groups are trivially word-hyperbolic.

The converse of Theorem 3.7 applies to nonelementary word-hyperbolic groups. In the proof of Theorem 3.2 we must separate the case where the group is elementary, since elementary groups may admit dominated representations while Theorem 3.7 does not apply to them.

3.3. Some preliminary lemmas for p-dominated representations

The *Grassmannian* $\mathscr{G}_p(\mathbb{R}^d)$ is the set of all p-dimensional subspaces of $\mathbb{R}^d$. As we explain in detail in the Appendix, the following formula defines a metric on the Grassmannian:

$$d(P, Q) := \cos \angle(P^\perp, Q);$$

here $\perp$ denotes orthogonal complement, and $\angle$ denotes the smallest angle between pairs of vectors in the respective spaces. The Appendix also contains a number of quantitative linear-algebraic estimates that we use in this section.

In particular, in the Appendix there are precise statements and proofs of the following results we will use:

- Lemmas A.4 and A.5 estimate the distance from $U_p(A)$ to $U_p(AB)$ and from $BU_p(A)$ to $U_p(BA)$ in terms of the norms of $B^{\pm 1}$ and the gaps between singular values of A. Lemma A.12 is a reinterpretation of these facts in terms of dominated sequences of matrices.
- Lemma A.6 shows that if there is a large gap between the p-th and $p + 1$-th singular values of A then AP will be close to $U_p(A)$ for every P which makes a given angle with $S_{d-p}(A)$.
- Lemma A.10 estimates the dilatation of the action of A in $\mathscr{G}_p(\mathbb{R}^d)$ for subspaces far from the subspace which is mapped to $U_p(A)$.
- Corollary A.14 is a consequence of classical properties of dominated splittings in the context of sequences of matrices in $\mathcal{D}(K, p, \mu, c, \mathbb{N})$.

Throughout the rest of this section, let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a p-dominated representation with constants $C \geq 1$ and $\lambda > 0$ (cf. (3.2)). Let

$$K := \max_{g \in S} \|\rho(g)\| \geq 1, \tag{3.3}$$

where S is the finite symmetric generating set of Γ fixed before. Also fix $\ell_0 \in \mathbb{N}$ such that

$$Ce^{-\lambda \ell_0} < 1, \tag{3.4}$$

and in particular (recalling the notation introduced in §2.2) the spaces $U_p(\rho(\gamma)) \in \mathscr{G}_p(\mathbb{R}^d)$ and $S_{d-p}(\rho(\gamma)) \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ are well-defined whenever $|\gamma| \geq \ell_0$.

Suppose that γ, η are large elements in Γ such that the spaces $U_p(\rho(\gamma))$ and $U_p(\rho(\eta))$ are not too close; then the next two lemmas respectively assert that $d(\gamma, \eta)$ is comparable to $|\gamma| + |\eta|$, and that $U_p(\rho(\gamma))$ and $S_{d-p}(\rho(\eta^{-1}))$ are transverse.

Lemma 3.9. *There exist constants $\nu \in (0, 1)$ and $c_0, c_1 > 0$ with the following properties. If $\gamma, \eta \in \Gamma$ are such that $|\gamma|, |\eta| \geq \ell_0$ (where ℓ_0 is as in (3.4)) then*

$$d(\gamma, \eta) \geq \nu(|\gamma| + |\eta|) - c_0 - c_1 \big|\log d\big(U_p(\rho(\gamma)), U_p(\rho(\eta))\big)\big|.$$

Proof. Let $\gamma, \eta \in \Gamma$, each with word-length at least ℓ_0. Assume $|\gamma| \leq |\eta|$, the other case being analogous. Applying Lemma A.4 to $A = \rho(\eta)$ and $B = \rho(\eta^{-1}\gamma)$, and using (3.2) and (3.3), we obtain

$$d\big(U_p(\rho(\eta)), U_p(\rho(\gamma))\big) \leq K^{2|\eta^{-1}\gamma|} Ce^{-\lambda|\eta|}, \tag{3.5}$$

or equivalently

$$d(\gamma, \eta) = |\eta^{-1}\gamma| \geq \frac{\lambda|\eta| - \log C - \big|\log d\big(U_p(\rho(\eta)), U_p(\rho(\gamma))\big)\big|}{2 \log K}.$$

Using $|\eta| \geq (|\gamma| + |\eta|)/2$, we obtain the lemma. $\qquad \square$

Lemma 3.10. *For every $\varepsilon > 0$ there exist $\ell_1 \geq \ell_0$ and $\delta \in (0, \pi/2)$ with the following properties: If $\gamma, \eta \in \Gamma$ are such that*

(i) $|\gamma|, |\eta| > \ell_1$, *and*
(ii) $d(U_p(\rho(\gamma)), U_p(\rho(\eta))) > \varepsilon$,

then $\angle(U_p(\rho(\gamma)), S_{d-p}(\rho(\eta^{-1}))) > \delta$.

Proof. Let $\gamma, \eta \in \Gamma$, each with word-length at least ℓ_0. Let $\varepsilon > 0$, and suppose the hypothesis (ii) is satisfied. Write $\gamma = g_1 \cdots g_n$ with each g_i in the symmetric generating set of Γ and with n minimal, that is, $n = |\gamma|$. Similarly, write $\eta = h_1 \cdots h_m$ with each h_i in the symmetric generating set of Γ and with $|\eta| = m$.

Let $\gamma_i := g_1 \cdots g_i$ and $\eta_i := h_1 \cdots h_i$. Note that for $j > i$ we have $d(\gamma_i, \gamma_j) = |\gamma_i^{-1}\gamma_j| = |g_{i+1} \cdots g_j| = j - i$ so that the sequence $\{\gamma_i\}$ is a geodesic from id to γ. The same holds for $\{\eta_i\}$ (but note that $\{\eta_i^{-1}\}$ is not necessarily a geodesic).

By the domination condition (3.2) and Lemma A.12, we can find $\ell_* = \ell_*(\varepsilon) > \ell_0$ such that if $n \geq i > \ell_*$ then

$$d\big(U_p(\rho(\gamma_i)), U_p(\rho(\gamma))\big) < \varepsilon/3,$$

and analogously if $m \geq j > \ell_*$ then

$$d\big(U_p(\rho(\eta_j)), U_p(\rho(\eta))\big) < \varepsilon/3.$$

In particular, if both conditions hold then hypothesis (ii) implies that

$$d\big(U_p(\rho(\gamma_i)), U_p(\rho(\eta_j))\big) > \varepsilon/3. \tag{3.6}$$

Let $\nu \in (0, 1)$ and c_0, c_1 be given by Lemma 3.9. Let $c := \max\{2\ell_*, c_0 + c_1 \log(3/\varepsilon)\}$. We claim that

$$d(\gamma_i, \eta_j) = |\eta_j^{-1}\gamma_i| \geq \nu(i + j) - c \quad \text{for all } i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}. \tag{3.7}$$

To prove this, consider first the case when $j \leq \ell_*$. Then

$$d(\gamma_i, \eta_j) \geq d(\gamma_i, \mathrm{id}) - d(\eta_j, \mathrm{id}) = i - j \geq i + j - 2\ell_* \geq \nu(i + j) - c,$$

as claimed. The case $i \leq \ell_*$ is dealt with analogously. The remaining case where i and j are both bigger than ℓ_* follows from Lemma 3.9 and property (3.6). This proves (3.7).

As a consequence of (3.2) and (3.7), we obtain

$$\frac{\sigma_{p+1}}{\sigma_p}(\eta_j^{-1}\gamma_i) \leq \hat{C}e^{-\mu(i+j)} \quad \text{for all } i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}, \tag{3.8}$$

where $\hat{C} := Ce^{\lambda c}$ and $\mu := \lambda\nu$. Now consider the following sequence of matrices:

$$(A_{-n}, \ldots, A_{-1}, A_0, A_1, \ldots, A_{m-1})$$
$$:= \big(\rho(g_n), \ldots, \rho(g_1), \rho(h_1^{-1}), \rho(h_2^{-1}), \ldots, \rho(h_m^{-1})\big).$$

So $A_{-1}A_{-2}\cdots A_{-i} = \rho(\gamma_i)$ and $A_{j-1}A_{j-2}\cdots A_0 = \rho(\eta_j^{-1})$. It follows from (3.2) and (3.8), together with the facts that $\hat{C} > C$ and $\mu < \lambda$, that the above sequence of matrices belongs to the set $\mathcal{D}(K, p, \mu, \hat{C}, I)$, where $I := \{-n, \dots, m-1\}$ and K is defined by (3.3). Now let ℓ_1 and δ be given by Lemma 2.5. Then

$$\angle\big(U_p(A_{-1}\cdots A_{-n}), S_{d-p}(A_{m-1}\cdots A_0)\big) > \delta$$

provided that $|\gamma| = n$ and $|\eta| = m$ are both bigger than ℓ_1. This concludes the proof. $\quad\square$

3.4. Candidate for boundary of Γ

In this subsection we define a candidate for M in Theorem 3.7 using the domination of the representation and show some of the topological properties of M required in Theorem 3.7. The set we shall consider is

$$M := \bigcap_{n \geq \ell_0} \overline{\{U_p(\rho(\gamma)) : |\gamma| \geq n\}} \subset \mathcal{G}_p(\mathbb{R}^d), \tag{3.9}$$

where ℓ_0 is as in (3.4). This set has been considered before and named *limit set* by Benoist (see [Be, Section 6]) in the Zariski dense context and extended in [GGKW, Definition 5.1] to a more general setting.

The first properties to be established about M are the following:

Proposition 3.11. *The set M is compact, nonempty, and $\rho(\Gamma)$-invariant.*

Proof. The fact that M is compact and nonempty follows at once since it is a decreasing intersection of nonempty closed subsets of the compact space $\mathcal{G}_p(\mathbb{R}^d)$.

Let us show that M is $\rho(\Gamma)$-invariant. Fix $\eta \in \Gamma$ and $P \in M$. Choose a sequence (γ_n) in Γ such that $|\gamma_n| \to \infty$ and $U_p(\rho(\gamma_n)) \to P$. Note that the spaces $U_p(\rho(\eta\gamma_n))$ are defined for large enough n (namely, whenever $|\gamma_n| \geq \ell_0 + |\eta|$); moreover, by Lemma A.5 and the domination condition (3.2), we have

$$d\big(\rho(\eta)U_p(\rho(\gamma_n)), U_p(\rho(\eta\gamma_n))\big) \leq \|\rho(\eta)\|\,\|\rho(\eta)^{-1}\|Ce^{-\lambda|\gamma_n|} \to 0 \quad \text{as } n \to \infty.$$

This shows that $U_p(\rho(\eta\gamma_n)) \to \rho(\eta)P$ as $n \to \infty$, and in particular $\rho(\eta)P \in M$, as we wished to prove. $\quad\square$

If one manages to show that $M^{(3)}$ is nonempty then one of the assumptions of Theorem 3.7 is satisfied:

Lemma 3.12. *If the set M has at least three points then it is perfect.*

We first show the following lemma:

Lemma 3.13. *Given $\varepsilon, \varepsilon' > 0$, there exists $\ell > \ell_0$ (where ℓ_0 is as in (3.4)) with the following properties: If $\eta \in \Gamma$ is such that $|\eta| > \ell$ and $P \in M$ is such that*

$$d\big(P, U_p(\rho(\eta^{-1}))\big) > \varepsilon,$$

then

$$d\big(\rho(\eta)P, U_p(\rho(\eta))\big) < \varepsilon'.$$

Proof. Let $\ell_1 \geq \ell_0$ and $\delta > 0$ be given by Lemma 3.10, depending on ε. Let $\ell > \ell_1$ be such that $Ce^{-\lambda\ell} < (\varepsilon' \sin \delta)/2$, where C and λ are the domination constants, as in (3.2). Now fix $\eta \in \Gamma$ and $P \in M$ such that $|\eta| > \ell$ and $d(P, U_p(\rho(\eta^{-1}))) > \varepsilon$. Choose a sequence (γ_n) in Γ such that $|\gamma_n| \to \infty$ and $U_p(\rho(\gamma_n)) \to P$. Without loss of generality, we can assume that for each n we have $|\gamma_n| > \ell_1$ and

$$d\big(U_p(\rho(\gamma_n)), U_p(\rho(\eta^{-1}))\big) > \varepsilon.$$

It follows from Lemma 3.10 that

$$\measuredangle\big(U_p(\rho(\gamma_n)), S_{d-p}(\rho(\eta))\big) > \delta.$$

Using Lemma A.6 and the domination condition (3.2), we obtain

$$d\big(\rho(\eta)(U_p(\rho(\gamma_n))), U_p(\rho(\eta))\big) < \frac{\sigma_{p+1}}{\sigma_p}(\rho(\eta)) \, \frac{1}{\sin \delta} < \frac{\varepsilon'}{2}.$$

Letting $n \to \infty$ yields $d(\rho(\eta)P, U_p(\rho(\eta))) \leq \varepsilon'/2$. $\qquad\square$

Proof of Lemma 3.12. Let P_1, P_2, P_3 be distinct points in M, and let $\varepsilon' > 0$. We will show that the $2\varepsilon'$-neighborhood of P_1 contains another element of M.

Let $\varepsilon := \frac{1}{2} \min_{i \neq j} d(P_i, P_j)$. Let $\ell > \ell_0$ be given by Lemma 3.13, depending on ε and ε'. Choose $\eta \in \Gamma$ such that $|\eta| > \ell$ and $d(U_p(\rho(\eta)), P_1) < \varepsilon'$. Consider the space $U_p(\rho(\eta^{-1}))$; it can be ε-close to at most one of the spaces P_1, P_2, P_3. In other words, there are different indices $i_1, i_2 \in \{1, 2, 3\}$ such that for each $j \in \{1, 2\}$ we have

$$d\big(P_{i_j}, U_p(\rho(\eta^{-1}))\big) > \varepsilon.$$

In particular, by Lemma 3.13,

$$d(\rho(\eta)P_{i_j}, P_1) \leq d\big(\rho(\eta)P_{i_j}, U_p(\rho(\eta))\big) + \varepsilon' < 2\varepsilon'.$$

By Proposition 3.11, the spaces $\rho(\eta)P_{i_1}$ and $\rho(\eta)P_{i_2}$ belong to M. Since at most one of them can be equal to P_1, we conclude that the $2\varepsilon'$-neighborhood of P_1 contains another element of M, as we wished to prove. $\qquad\square$

So we would like to ensure that M has at least three points (provided Γ is nonelementary). This requires some preliminaries. First, we show the following estimate:

Lemma 3.14. *M has at least two elements.*

Proof. Using Lemma 2.5 we can find $\ell_1 \in \mathbb{N}$ and $\delta > 0$ such that if $\gamma, \eta \in \Gamma$ have lengths both bigger than ℓ_1, and $|\eta\gamma| = |\eta| + |\gamma|$, then

$$\measuredangle\big(U_p(\rho(\gamma)), S_{d-p}(\rho(\eta))\big) > \delta. \tag{3.10}$$

We already know that M is nonempty, so assume for a contradiction that it is a singleton $\{P\}$. Take $n > \ell_1$ such that if an element $\gamma \in \Gamma$ has length $|\gamma| > n$ then $d(U_p(\rho(\gamma)), P) < \frac{1}{2} \sin \delta$. Now fix arbitrary elements $\gamma, \eta \in \Gamma$ with lengths both bigger than ℓ_1 such that $|\eta\gamma| = |\eta| + |\gamma|$. Then $d(U_p(\rho(\gamma)), U_p(\rho(\eta^{-1}))) < \sin \delta$. The space $U_p(\rho(\eta^{-1}))$ either contains or is contained in $U_{d-p}(\rho(\eta^{-1})) = S_{d-p}(\rho(\eta))$. Using the trivial bound (A.4), we contradict (3.10). $\qquad\square$

Recalling Remark 3.3, assume for convenience for the remainder of this subsection that

$$|\det \rho(\gamma)| = 1 \quad \text{for every } \gamma \in \Gamma.$$

Let $\mathrm{jac}(\cdot)$ denote the *jacobian* of a linear map, i.e., the product of its singular values. The next step is to prove the expansion property expressed by the following lemma:

Lemma 3.15. *There exists a constant $\ell \in \mathbb{N}$ such that for every $\gamma \in \Gamma$ with length $|\gamma| \geq \ell$ there exists $P \in M$ such that $\mathrm{jac}(\rho(\gamma)|_P) \geq 2$.*

Proof. As a rephrasing of Lemma 3.14, there exists $\varepsilon > 0$ such that for each $Q \in M$ there exists $P \in M$ such that $d(P, Q) > 2\varepsilon$. Let $\ell_1 \geq \ell_0$ and $\delta \in (0, \pi/2)$ be given by Lemma 3.10, depending on ε. Let

$$\delta' := \min \tfrac{1}{2}\{\varepsilon, 1 - \cos \delta\}.$$

Let $\ell_2 \geq \ell_1$ be such that the sets M and $\{U_p(\rho(\gamma)) : \gamma \in \Gamma, \, |\gamma| \geq \ell\}$ are each contained in the δ'-neighborhood of the other. Fix a large $\ell \geq \ell_2$; how large will be clear at the end.

Now fix $\gamma \in \Gamma$ with length $|\gamma| \geq \ell$. Take $Q \in M$ such that $d(Q, U_p(\rho(\gamma^{-1}))) < \delta'$. Let $P \in M$ with $d(P, Q) > 2\varepsilon$. Let $\eta \in \Gamma$ be such that $|\eta| \geq \ell$ and $d(P, U_p(\rho(\eta))) < \delta'$. Then

$$\begin{aligned}
d\big(U_p(\rho(\eta)), U_p(\rho(\gamma^{-1}))\big) &\geq d(P, Q) - d\big(U_p(\rho(\eta)), P\big) - d\big(Q, U_p(\rho(\gamma^{-1}))\big) \\
&> 2\varepsilon - 2\delta' > \varepsilon.
\end{aligned}$$

Therefore Lemma 3.10 guarantees that

$$\angle\big(U_p(\rho(\eta)), S_{d-p}(\rho(\gamma))\big) > \delta.$$

So, by identity (A.8),

$$\begin{aligned}
\cos \angle\big(P, S_{d-p}(\rho(\gamma))\big) &= d\big(P, S_{d-p}(\rho(\gamma))^{\perp}\big) \\
&\leq d\big(P, U_p(\rho(\eta))\big) + d\big(U_p(\rho(\eta)), S_{d-p}(\rho(\gamma))^{\perp}\big) \\
&= d\big(P, U_p(\rho(\eta))\big) + \cos \angle\big(U_p(\rho(\eta)), S_{d-p}(\rho(\gamma))\big) \\
&\leq \delta' + \cos \delta < 1.
\end{aligned}$$

Write $A := \rho(\gamma)$ for simplicity. We must estimate the jacobian of $A|_P$. We will use some facts about exterior powers: see Subsection A.5. Let $\iota(P) \in \mathscr{G}_1(\Lambda^p \mathbb{R}^d)$ be the image of P under the Plücker embedding, and take a nonzero $w \in \iota(P)$. Then

$$\begin{aligned}
\mathrm{jac}(A|_P) &= \frac{\|(\Lambda^p A)w\|}{\|w\|} && \text{(by (A.16))} \\
&\geq \sigma_1(\Lambda^p A) \sin \angle\big(w, S_{\binom{d}{p}-1}(\Lambda^p A)\big) && \text{(by (A.11))} \\
&= \sigma_1(A) \cdots \sigma_p(A) \sin \angle\big(\iota(P), \iota(S_{d-p}(A^{\perp})^{\perp})\big) && \text{(by (A.19) and (A.22))} \\
&\geq \sigma_1(A) \cdots \sigma_p(A)[\sin \angle(P, S_{d-p}(A))]^{\min\{p, d-p\}} && \text{(by (A.18))}
\end{aligned}$$

The sine can be bounded from below by a positive constant. On the other hand, since the product of the singular values of A is $|\det A| = 1$, we have

$$\sigma_1(A) \cdots \sigma_p(A) = \frac{[\sigma_1(A) \cdots \sigma_p(A)]^{(d-p)/d}}{[\sigma_{p+1}(A) \cdots \sigma_d(A)]^{p/d}} \geq \left[\frac{\sigma_p(A)}{\sigma_{p+1}(A)} \right]^{p(d-p)/d},$$

which is exponentially large with respect to $|\gamma| \geq \ell$. We conclude that $\mathrm{jac}(A|_P) \geq 2$ if ℓ is large enough. $\qquad\square$

As a last digression, we show that virtually abelian groups cannot have a dominated representation unless they are virtually cyclic.

Lemma 3.16. *Let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a p-dominated representation. Let Γ' be a finite-index subgroup of Γ, and let $m \geq 2$. Then there exists no surjective homomorphism $\varphi \colon \Gamma' \to \mathbb{Z}^m$ with finite kernel.*

Proof. Assume for a contradiction that Γ contains a subgroup Γ' which admits a homomorphism φ onto $\mathbb{Z}^m$ with finite kernel, where $m \geq 2$. Let Z be the standard symmetric generating set for $\mathbb{Z}^m$, with cardinality $2m$; then $\varphi^{-1}(Z)$ is a finite symmetric generating set for Γ'. Since Γ' is a finite-index subgroup of Γ, the inclusion $\Gamma' \hookrightarrow \Gamma$ is a quasi-isometry (using e.g. the Švarc–Milnor lemma [BH, Proposition I.8.19]), and therefore the restriction of ρ to Γ' is p-dominated. For simplicity of notation, we assume that $\Gamma = \Gamma'$, with generating set $S = \varphi^{-1}(Z)$. Fix constants C, λ, K, and ℓ_0 satisfying (3.2)–(3.4).

Fix $g \in \Gamma$ such that $\varphi(g) \in Z$, say $(1, 0, \ldots, 0)$. Since the representation is p-dominated and $(g^n)_{n \in \mathbb{Z}}$ is a geodesic in Γ, it follows from Lemma 2.5 that there exist $\ell_1 > \ell_0$ and $\delta > 0$ such that

$$\angle\big(U_p(\rho(g^n)), S_{d-p}(\rho(g^n))\big) > \delta \quad \text{for all } n > \ell_1.$$

The space $S_{d-p}(\rho(g^n)) = U_{d-p}(\rho(g^{-n}))$ either contains or is contained in $U_p(\rho(g^{-n}))$; so the trivial bound (A.4) yields

$$d\big(U_p(\rho(g^n)), U_p(\rho(g^{-n}))\big) \geq \sin \delta \quad \text{for all } n > \ell_1. \qquad (3.11)$$

Fix $n > \ell_1$. Since $m \geq 2$, we can find $\gamma_0, \gamma_1, \ldots, \gamma_{4n} \in \Gamma$ with the following properties:

$$\gamma_0 = g^n, \quad \gamma_{4n} = g^{-n}, \quad |\gamma_{i+1}^{-1} \gamma_i| = 1, \quad |\gamma_i| \geq n.$$

Indeed, we can take a preimage under φ of an appropriate path in $\mathbb{Z}^m$, sketched in Fig. 3 for the case $m = 2$.
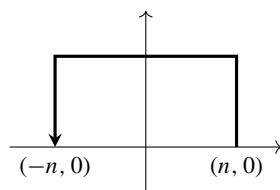

Fig. 3. A path in $\mathbb{Z}^2$.

Now we estimate

$$
\begin{aligned}
d\big(U_p(\rho(g^n)), U_p(\rho(g^{-n}))\big) &\leq \sum_{i=0}^{4n-1} d\big(U_p(\rho(\gamma_i)), U_p(\rho(\gamma_{i+1}))\big) \\
&\leq \sum_{i=0}^{4n-1} K^{2|\gamma_{i+1}^{-1}\gamma_i|} C e^{-\lambda|\gamma_i|} \quad \text{(by estimate (3.5))} \\
&\leq 4n K^2 C e^{-\lambda n}.
\end{aligned}
$$

Taking n large enough, we contradict (3.11). This proves the lemma. $\qquad\square$

Now we are ready to obtain the last topological property of M required by Theorem 3.7. (Later we will have to check the hypotheses about the action on $M^{(3)}$.) Recall that Γ is elementary if it is finite or virtually cyclic (i.e. virtually $\mathbb{Z}$).

Proposition 3.17. *If Γ is nonelementary then the set M is perfect.*

Proof. By Lemma 3.12, it is enough to show that M is infinite. Recall that M is nonempty. We assume for a contradiction that M is finite, say $M = \{P_1, \ldots, P_k\}$.

By Proposition 3.11, the set M is $\rho(\Gamma)$-invariant. Consider the set $\Gamma' \subset \Gamma$ of those elements such that $\rho(\gamma) P_i = P_i$ for all i. Then Γ' is a finite-index subgroup of Γ. Consider the map $\varphi \colon \Gamma' \to \mathbb{R}^k$ defined by

$$
\Gamma' \ni \gamma \mapsto (\log \mathrm{jac}(\rho(\gamma)|_{P_i}))_{i=1,\ldots,k}.
$$

Invariance of the subspaces P_i implies that the jacobian is multiplicative and so φ is a homomorphism. By Lemma 3.15, the kernel of φ is finite. On the other hand, the image of φ is a subgroup of $\mathbb{R}^k$ and therefore is abelian, without torsion, and finitely generated (since so is Γ'); therefore this image is isomorphic to some $\mathbb{Z}^m$. Lemma 3.16 now yields $m = 1$, and so Γ is elementary. This contradiction concludes the proof. $\qquad\square$

3.5. Proper discontinuity

Given a triple $T = (P_1, P_2, P_3) \in M^{(3)}$, let

$$
|T| := \min_{i \neq j} d(P_i, P_j). \tag{3.12}
$$

Note that for any $\delta > 0$, the set $\{T \in M^{(3)} : |T| \geq \delta\}$ is a compact subset of $M^{(3)}$; conversely, every compact subset of $M^{(3)}$ is contained in a subset of that form.

Proposition 3.18. *For every $\delta > 0$ there exists $\ell \in \mathbb{N}$ such that if $T \in M^{(3)}$ satisfies $|T| > \delta$ and $\eta \in \Gamma$ satisfies $|\eta| > \ell$, then $|\rho(\eta)T| < \delta$.*

Proof. Given $\delta > 0$, let ℓ be given by Lemma 3.13 with $\varepsilon = \varepsilon' = \delta/2$. Now consider $(P_1, P_2, P_3) \in M^{(3)}$ such that $|T| > \delta$ and $\eta \in \Gamma$ such that $|\eta| > \ell$. Note that $d(U_p(\eta^{-1}), P_i) > \delta/2$ for at least two of the spaces P_1, P_2, P_3, say P_1 and P_2. Lemma 3.13 yields $d(\rho(\eta)P_i, U_p(\rho(\eta))) < \delta/2$ for each $i = 1, 2$. In particular, $d(\rho(\eta)P_1, \rho(\eta)P_2) < \delta$ and so $|\rho(\eta)T| < \delta$, as we wanted to show. $\qquad\square$

3.6. Cocompactness

The purpose of this subsection is to prove the following proposition which will complete the proof of Theorem 3.2. Recall notation (3.12).

Proposition 3.19. *There exists $\varepsilon > 0$ such that for every $T \in M^{(3)}$ there exists $\gamma \in \Gamma$ such that $|\rho(\gamma)T| \geq \varepsilon$.*

We need some preliminaries. Recall that a *geodesic ray* from the identity is a sequence $(\eta_j)_{j \geq 0}$ such that $\eta_0 = \mathrm{id}$, $\eta_j = g_1 \cdots g_j$ where each g_j belongs to S (the fixed symmetric generating set of Γ), and $|\eta_j| = j$. Then we have the following characterization of M:

Lemma 3.20. *For every $P \in M$ there exists a geodesic ray $(\eta_j)_{j \geq 0}$ from the identity such that $U_p(\rho(\eta_j)) \to P$ as $j \to \infty$.*

Proof. Fix $P \in M$. By definition, there exists a sequence (γ_i) in Γ with $n_i := |\gamma_i| \to \infty$ such that $U_p(\rho(\gamma_i)) \to P$. We can assume that $n_1 < n_2 < \cdots$. Write each γ_i as a product of elements of S, say $\gamma_i = g_1^{(i)} g_2^{(i)} \cdots g_{n_i}^{(i)}$. By a diagonal argument, we can assume that each of the sequences $(g_j^{(i)})_i$ stabilizes; more precisely, for every $j \geq 1$ there exist $g_j \in S$ and k_j such that $n_{k_j} \geq j$ and $g_j^{(i)} = g_j$ for every $i \geq k_j$.

Consider the geodesic ray (η_j) defined by $\eta_j := g_1 \cdots g_j$. Let $i_j := \max\{k_1, \ldots, k_j\}$. We will prove that

$$d\big(U_p(\rho(\eta_j)), U_p(\rho(\gamma_{i_j}))\big) \to 0, \tag{3.13}$$

which has the desired property $U_p(\rho(\eta_j)) \to P$ as a consequence.

Consider the truncated products $\gamma_i^{[j]} := g_1^{(i)} g_2^{(i)} \cdots g_j^{(i)}$. Using Lemma A.4 and the fact that the representation ρ is p-dominated (or equivalently using estimate (3.5)), we see that $d(U_p(\rho(\gamma_i^{[j]})), U_p(\rho(\gamma_i^{[j+1]})))$ can be bounded by a quantity exponentially small with respect to j and independent of i. Therefore $d(U_p(\rho(\gamma_i^{[j]})), U_p(\rho(\gamma_i)))$ is also exponentially small with respect to j. Applying this to $i = i_j$, we obtain (3.13). $\square$

Lemma 3.21 (Expansivity). *There exist constants $\delta > 0$ and $\ell \in \mathbb{N}$ with the following properties. For every $P \in M$ there exists $\gamma \in \Gamma$ with $|\gamma| \leq \ell$ such that if P', P'' belong to the δ-neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ then*

$$d(\rho(\gamma)P', \rho(\gamma)P'') \geq 2d(P', P'').$$

Proof. By compactness of M, it is sufficient to prove that for every $P \in M$ there exists $\gamma \in \Gamma$ such that if P', P'' belong to a sufficiently small neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ then $d(\rho(\gamma)P', \rho(\gamma)P'') \geq 2d(P', P'')$.

Given $P \in M$, by Lemma 3.20 there exists a geodesic ray (η_n) such that $U_p(\rho(\eta_n)) \to P$. Write $\eta_n = g_1 \cdots g_n$, where each g_n is in S. Consider the sequence of matrices $(A_0, A_1, \ldots)$ given by $A_n := \rho(g_{n-1}^{-1})$. By the domination condition (3.2), the sequence belongs to some $\mathcal{D}(\mu, c, K, d - p, \mathbb{N})$. Note that $S_p(A_{n-1} \cdots A_0)$ equals $U_p(\rho(\eta_n))$ and

therefore converges to P. Applying Corollary A.14 to the sequence of matrices, we find $\tilde{P} \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ such that, for all $n \geq 0$,

$$\angle(\rho(\eta_n^{-1})\tilde{P}, \rho(\eta_n^{-1})P) \geq \alpha, \qquad \frac{\|\rho(\eta_n^{-1})|_P\|}{\mathbf{m}(\rho(\eta_n^{-1})|_{\tilde{P}})} < \tilde{c}e^{-\tilde{\mu}n},$$

where α, $\tilde{c}$, $\tilde{\mu}$ are positive constants that do not depend on P. Let $b > 0$ be given by Lemma A.10, depending on α. Fix k such that $b\tilde{c}^{-1}e^{\tilde{\mu}k} > 2$, and let $\gamma := \eta_k^{-1}$. Applying Lemma A.10 to $A := \rho(\gamma)$, we conclude that for all P', P'' in a sufficiently small neighborhood of P in $\mathscr{G}_p(\mathbb{R}^d)$ we have

$$d(\rho(\gamma)P', \rho(\gamma)P'') \geq 2d(P', P''),$$

as we wanted to show. $\square$

Proof of Proposition 3.19. Let δ and ℓ be given by Lemma 3.21. Let

$$\varepsilon := \inf\{d(\rho(\gamma)P, \rho(\gamma)P') : \gamma \in \Gamma, \ |\gamma| \leq \ell, \ P, P' \in \mathscr{G}_p(\mathbb{R}^d), \ d(P, P') \geq \delta/2\}.$$

So $0 < \varepsilon \leq \delta/2$. We claim that

$$\forall T \in M^{(3)} \ \exists \gamma \in \Gamma, \quad |\rho(\gamma)T| \geq \min\{2|T|, \varepsilon\}. \tag{3.14}$$

Indeed, given $T = (P_1, P_2, P_3) \in M^{(3)}$, we can suppose that $|T| < \varepsilon$, otherwise we simply take $\gamma = \mathrm{id}$. Permuting indices if necessary we can assume that $d(P_1, P_2) = |T|$. We apply Lemma 3.21 and find $\gamma \in \Gamma$ such that the action of $\rho(\gamma)$ on $N_\delta(P_1)$ (the δ-neighborhood of P_1) expands distances by a factor of at least 2. Since $\varepsilon \leq \delta/2$, for each pair $\{i \neq j\} \subset \{1, 2, 3\}$ we have

$$\{P_i, P_j\} \subset N_\delta(P_1) \quad \text{or} \quad d(P_i, P_j) \geq \delta/2.$$

So $d(\rho(\gamma)P_i, \rho(\gamma)P_j) \geq \min\{2|T|, \varepsilon\}$, thus proving the claim (3.14). Now the proposition follows by an obvious recursive argument. $\square$

3.7. Conclusion

Now we join the pieces and obtain the main result of this section:

Proof of Theorem 3.2. Consider a p-dominated representation $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$. If Γ is an elementary group then it is word-hyperbolic and there is nothing to prove. So assume that Γ is nonelementary. Using the representation ρ, we define an action of Γ on $\mathscr{G}_p(\mathbb{R}^d)$. Consider the set $M \subset \mathscr{G}_p(\mathbb{R}^d)$ defined by (3.9), which is perfect (by Proposition 3.17) and invariant under the action of Γ (by Proposition 3.11). The diagonal action of Γ on $M^{(3)}$ is properly discontinuous (by Proposition 3.18) and cocompact (by Proposition 3.19). Therefore Theorem 3.7 ensures that Γ is word-hyperbolic. $\square$

4. Anosov representations and dominated representations

The main goal of this section is to show that being p-dominated (cf. condition (3.2)) and satisfying the Anosov condition as defined by Labourie [Lab] (and extended by Guichard–Wienhard [GW] to arbitrary hyperbolic groups) are equivalent. That equivalence (among

others things) is contained in the results of [KLP$_1$, KLP$_2$, KLP$_3$]. Our approach also yields a slightly different characterization directly related to dominated splittings (see Proposition 4.6).

In the final subsection we discuss relations to characterizations of [GGKW], and pose some questions.

We first introduce the notion of Anosov representation in $GL(d, \mathbb{R})$, which requires introducing the geodesic flow of a hyperbolic group.

4.1. The geodesic flow

In order to define the Anosov property for a representation of a hyperbolic group, we need to recall the *Gromov geodesic flow* of Γ.

Given a word-hyperbolic group Γ we can define its visual boundary $\partial\Gamma$ (cf. Remark 3.8). Denote $\partial^{(2)}\Gamma := \{(x, y) \in \partial\Gamma \times \partial\Gamma : x \neq y\}$. We define a flow on the space $\widetilde{U\Gamma} := \partial^{(2)}\Gamma \times \mathbb{R}$, called the *lifted geodesic flow*, by $\tilde{\phi}^t(x, y, s) := (x, y, s + t)$.

A function $c : \Gamma \times \partial^{(2)}\Gamma \to \mathbb{R}$ such that

$$c(\gamma_0\gamma_1, x, y) = c(\gamma_0, \gamma_1(x, y)) + c(\gamma_1, x, y) \quad \text{for any } \gamma_0, \gamma_1 \in \Gamma \text{ and } (x, y) \in \partial^{(2)}\Gamma$$

is called a *cocycle*. Recall that every infinite order element $\gamma \in \Gamma$ acts on $\partial\Gamma$ leaving only two fixed points, an attractor γ^+ and a repeller γ^-. Let us say that a cocycle c is *positive* if $c(\gamma, \gamma^-, \gamma^+) > 0$ for every such γ.

Given a cocycle, we can define an action of Γ on $\widetilde{U\Gamma}$ by $\gamma \cdot (x, y, s) = (\gamma \cdot x, \gamma \cdot y, s - c(x, y, \gamma))$, which obviously commutes with the lifted geodesic flow. Gromov [Gr] (see also [Mat, Ch, Mi]) proved that there exists a positive cocycle such that the latter action is properly discontinuous and cocompact. This allows us to define the *geodesic flow* ϕ^t of Γ on $U\Gamma := \widetilde{U\Gamma}/\Gamma$, the *unit tangent bundle* of Γ.

There is a metric on $\widetilde{U\Gamma}$, well-defined up to Hölder equivalence, such that Γ acts by isometries, the lifted geodesic flow acts by bi-Lipschitz homeomorphisms, and its flow lines are quasi-geodesics.

Remark 4.1. If $\Gamma = \pi_1(M)$ is the fundamental group of a negatively curved closed manifold M then the geodesic flow on the unit tangent bundle UM is hyperbolic and equivalent to the abstract geodesic flow defined above. In that case, the unit tangent bundle of the universal cover $\tilde{M}$ is homeomorphic to $\widetilde{U\Gamma}$ by means of the Hopf parametrization. For details, see [Led].

Lemma 4.2. *For any compact set $K \subset \widetilde{U\Gamma}$, there exist $a > 0$ and $\kappa > 1$ such that if $t \in \mathbb{R}$ and $\gamma \in \Gamma$ satisfy*

$$\tilde{\phi}^t(K) \cap \gamma(K) \neq \emptyset$$

then

$$\kappa^{-1}|t| - a \leq |\gamma| \leq \kappa|t| + a.$$

Proof. Take a ball $B(u_0, r)$ containing K. From the construction (see e.g. [Mat, Theorem IV.1(ii)] or [Mi]) of the metric in $\widetilde{U\Gamma}$ one deduces that the map $\Gamma \ni \gamma \mapsto \gamma u_0 \in \widetilde{U\Gamma}$ is a

quasi-isometry, so there exist $\kappa > 1$ and $b > 0$ such that for all $\gamma_1, \gamma_2 \in \Gamma$ we have

$$\kappa^{-1} d(\gamma_1, \gamma_2) - b \le d(\gamma_1 u_0, \gamma_2 u_0) \le \kappa d(\gamma_1, \gamma_2) + b.$$

In particular,

$$\kappa^{-1} |\gamma| - b \le d(\gamma u_0, u_0) \le \kappa |\gamma| + b.$$

Now assume that t and γ satisfy $\tilde{\phi}^t(K) \cap \gamma(K) \ne \emptyset$, that is, there exist $u_1, u_2 \in K$ such that $\tilde{\phi}^t u_1 = \gamma u_2$. So[6]

$$|t| \overset{*}{=} d(\tilde{\phi}^t u_1, u_1) = d(\gamma u_2, u_1) \begin{cases} \le d(\gamma u_0, u_0) + 2r, \\ \ge d(\gamma u_0, u_0) - 2r. \end{cases}$$

The desired inequalities follow. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

4.2. Equivariant maps and the definition of Anosov representations

Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation of a word-hyperbolic group Γ. The definitions here can be adapted to representations in general semisimple Lie groups and the results are equivalent. In order to be able to present our results in a more elementary manner, we have deferred the introduction of the general context to Section 8.

We say that the representation ρ is *p-convex* if there exist continuous maps $\xi : \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$ such that:

- (transversality) for all $x \ne y \in \partial\Gamma$ we have $\xi(x) \oplus \theta(y) = \mathbb{R}^d$,
- (equivariance) for every $\gamma \in \Gamma$ we have $\xi(\gamma \cdot x) = \rho(\gamma)\xi(x)$ and $\theta(\gamma \cdot x) = \rho(\gamma)\theta(x)$.

Using the representation ρ, it is possible to construct a linear flow ψ^t over the geodesic flow ϕ^t of Γ as follows. Consider the lifted geodesic flow $\tilde{\phi}^t$ on $\widetilde{U\Gamma}$, and define a linear flow on $\tilde{E} := \widetilde{U\Gamma} \times \mathbb{R}^d$ by

$$\tilde{\psi}^t((x, y, s), v) := (\tilde{\phi}^t(x, y, s), v),$$

Now consider the action of Γ on $\tilde{E}$ given by

$$\gamma \cdot ((x, y, s), v) := (\gamma \cdot (x, y, s), \rho(\gamma)v)$$

where the action of Γ in $\widetilde{U\Gamma}$ is the one explained in Subsection 4.1. It follows that $\tilde{\psi}^t$ induces in $E_\rho := \tilde{E}/\Gamma$ (which is a vector bundle over $U\Gamma$) a linear flow ψ^t which covers ϕ^t. See Fig. 4.

When the representation ρ is *p*-convex, by equivariance there exists a ψ^t-invariant splitting of the form $E_\rho = \Xi \oplus \Theta$; it is obtained by taking the quotients of the bundles $\tilde{\Xi}(x, y, s) := \xi(x)$ and $\tilde{\Theta}(x, y, s) := \theta(y)$ with respect to the Γ-action.

We say that a *p*-convex representation ρ is *p-Anosov* if the splitting $E_\rho = \Xi \oplus \Theta$ is a dominated splitting for the linear bundle automorphism ψ^t, with Ξ dominating Θ. This is equivalent to the bundle $\mathrm{Hom}(\Theta, \Xi)$ being uniformly contracted by the flow induced by ψ^t (see [BCLS]).

---

[6] To get exact equality in $(*)$ we need the construction in [Mi], for which orbits of the flow are geodesics. If instead we use [Ch] or [Mat], the orbits of the flow are quasi-geodesics, so the equality $(*)$ is only approximate, which is sufficient for our purpose.
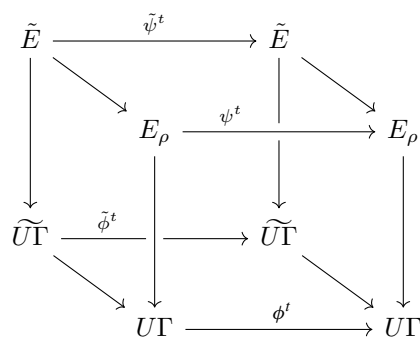
Fig. 4. A commutative diagram. The $\downarrow$ arrows are vector bundle projections. The $\searrow$ arrows are quotient maps with respect to the corresponding actions of Γ. The $\rightarrow$ arrows are the flow actions, as indicated.

Conversely, dominated splittings for the linear flow ψ^t must be of the form $\Xi \oplus \Theta$ as above: see Propositions 4.6 and 4.9 below.

Let us mention that by [GW, Theorem 1.5], if the image of the representation ρ is Zariski dense, then being p-Anosov is a direct consequence of being p-convex.

Remark 4.3. As before, it is possible to use exterior powers to transform a p-Anosov representation into a 1-Anosov one. The latter are called *projective Anosov* and are discussed in Section 8 where it is shown that any Anosov representation in an arbitrary semisimple Lie group can be transformed into a projective Anosov one. See also [BCLS, Section 2.3].

4.3. Equivalence between the definitions

We will show that a representation $\rho : \Gamma \rightarrow \mathrm{GL}(d, \mathbb{R})$ is p-dominated if and only if it is p-Anosov. Note that the definition of a p-Anosov representation requires the group to be word-hyperbolic. On the other hand, we have shown in Section 3 that if ρ is p-dominated then the group Γ is automatically word-hyperbolic. So we can assume in what follows that Γ is word-hyperbolic.

Let us first show the following:

Lemma 4.4. *Endow E_ρ with a Riemannian metric.*[7] *Then there exist constants $\kappa > 1$, $a > 0$, and $C > 1$ with the following properties:*

(i) *For every $z \in U\Gamma$ and $t \in \mathbb{R}$, there exists $\gamma \in \Gamma$ such that*

$$\kappa^{-1}|t| - a \leq |\gamma| \leq \kappa|t| + a, \tag{4.1}$$

$$C^{-1}\sigma_p(\psi_z^t) \leq \sigma_p(\rho(\gamma)) \leq C\sigma_p(\psi_z^t) \quad \text{for every } p = 1, \ldots, d - 1. \tag{4.2}$$

(ii) *Conversely, for every $\gamma \in \Gamma$ there exist $z \in U\Gamma$ and $t \in \mathbb{R}$ such that (4.1) and (4.2) hold.*

---

[7] The Riemannian metric allows us to consider singular values for the linear maps ψ_x^t.

Proof. Using the covering map $\tilde{E} \to E_\rho$, we lift the fixed Riemannian metric on E_ρ, obtaining a Riemannian metric $\|\cdot\|_*$ on $\tilde{E}$ preserved by the action of Γ. On the other hand, since the vector bundle $\tilde{E}$ is trivial, we can also endow it with the Euclidean metric $\|\cdot\|$ on the fibers. Let $K \subset \widetilde{U\Gamma}$ be a compact set intersecting every Γ-orbit. Then there exists $C_K > 1$ such that, for every $v \in \tilde{E}$ that projects on K,

$$C_K^{-1}\|v\| \leq \|v\|_* \leq C_K\|v\|.$$

By Lemma A.2, a bounded change of inner product has a bounded effect on the singular values. It follows that for all $\tilde{z} \in K$, $t \in \mathbb{R}$, and $\gamma \in \Gamma$ such that $\tilde{\phi}^t(\tilde{z}) \in \gamma(K)$, if z is the projection of $\tilde{z}$ in $U\Gamma$, inequality (4.2) holds for $C = C_K^2$.

The rest of part (i) follows directly from Lemma 4.2.

Now let us prove part (ii). Consider the action of Γ on the compact metric space $\partial\Gamma$. Fix a positive $\delta < \frac{1}{4}\text{diam}\,\partial\Gamma$.

Claim. *For every $\gamma \in \Gamma$ there exist $x, y \in \partial\Gamma$ such that $d(x, y) \geq \delta$ and $d(\gamma^{-1}x, \gamma^{-1}y) \geq \delta$.*

Proof of Claim. Fix $x_1, x_2 \in \partial\Gamma$ such that $d(x_1, x_2) > 3\delta$. We can assume that $d(\gamma^{-1}x_1, \gamma^{-1}x_2) < \delta$, because otherwise we would take $(x, y) := (x_1, x_2)$. It follows that there exists $i \in \{1, 2\}$ such that $d(x_i, \gamma^{-1}x_1), d(x_i, \gamma^{-1}x_2) > \delta$. Analogously, we can assume that $d(\gamma x_1, \gamma x_2) < \delta$, because otherwise we would take $(x, y) := (\gamma x_1, \gamma x_2)$. It follows that there exists $j \in \{1, 2\}$ such that $d(x_j, \gamma x_1), d(x_j, \gamma x_2) > \delta$. Therefore the pair $(x, y) := (x_j, \gamma x_i)$ has the desired properties. $\quad\square$

Let $K := \{(x, y, 0) : x, y \in \partial\Gamma, d(x, y) \geq \delta\}$; this is a compact subset of $\widetilde{U\Gamma}$. Therefore, letting $\tilde{z} := (x, y, 0)$ and $t := c(x, y, \gamma)$, we conclude as above. $\quad\square$

Now let us prove the equivalence between being p-dominated and p-Anosov. We first show:

Proposition 4.5. *Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a p-Anosov representation. Then ρ is p-dominated.*

Proof. Since ρ is p-Anosov, Theorem 2.2 implies that there exist $C, \lambda > 0$ such that for every $z \in U\Gamma$ and $t > 0$ we have

$$\frac{\sigma_{p+1}(\psi_z^t)}{\sigma_p(\psi_z^t)} < Ce^{-\lambda t}.$$

Using Lemma 4.4(i), we can find constants $C', \lambda' > 0$ such that for every $\gamma \in \Gamma$,

$$\frac{\sigma_{p+1}(\rho(\gamma))}{\sigma_p(\rho(\gamma))} < C'e^{-\lambda'|\gamma|}.$$

This means that ρ is p-dominated. $\quad\square$

Note that the proof above only uses the fact that the linear flow ψ^t has a dominated splitting, so we obtain:

Proposition 4.6. *If the linear flow ψ^t on E_ρ has a dominated splitting with dominating bundle of dimension p then ρ is p-dominated.*

To prove that p-domination implies being p-Anosov, we shall first show the existence of the equivariant maps ξ, θ. This is a relatively easy consequence of what is done in Section 3 (see Remark 3.8). The equivariant maps exist under an even weaker hypothesis, as shown in [GGKW, Theorem 5.2]. We provide a proof here for completeness.

Recall that an (a, b)-*quasi-geodesic* in Γ is a sequence $\{\gamma_n\}$ such that

$$a^{-1}|n - m| - b < d(\gamma_n, \gamma_m) < a|n - m| + b.$$

We denote by $\mathbb{Q}^{\mathrm{id}}_{(a,b)}$ the set of (a, b)-quasi-geodesics such that $\gamma_0 = \mathrm{id}$.

Lemma 4.7. *Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation such that for some $a, b > 0$,*

$$\sup_{\gamma_n \in \mathbb{Q}^{\mathrm{id}}_{(a,b)}} \sum_{n \geq n_0} \frac{\sigma_{p+1}(\rho(\gamma_n))}{\sigma_p(\rho(\gamma_n))} \xrightarrow[n_0 \to \infty]{} 0. \tag{4.3}$$

Then there exists an equivariant continuous map $\xi : \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ defined by

$$\xi(x) := \lim_n U_p(\rho(\gamma_n)),$$

where $\{\gamma_n\}$ is any (a, b)-quasi-geodesic ray representing $x \in \partial\Gamma$.

Proof. For each $x \in \partial\Gamma$, choose $\{\gamma_n^x\} \in \mathbb{Q}^{\mathrm{id}}_{(a,b)}$ representing x. We define

$$\xi(x) := \lim_n U_p(\rho(\gamma_n^x)).$$

To see that this limit exists, let C_0 be an upper bound of $\|\rho(g)\| \|\rho(g^{-1})\|$ for $g \in S$ a finite generating set of Γ and use Lemma A.4 to see that

$$d\big(U_p(\rho(\gamma_n^x)), U_p(\rho(\gamma_{n-1}^x))\big) \leq C_0^{d(\gamma_n, \gamma_{n-1})} \frac{\sigma_{p+1}(\rho(\gamma_{n-1}^x))}{\sigma_p(\rho(\gamma_{n-1}^x))}.$$

This implies that $U_p(\rho(\gamma_n^x))$ is a Cauchy sequence and therefore has a limit. The fact that the limit does not depend on the chosen (a, b)-quasi-geodesic follows directly from a similar estimate using Lemma A.4.

Since the estimates are uniform, this becomes a uniform limit as one changes $x \in \partial\Gamma$, providing continuity of the maps (recall the topology in $\partial\Gamma$ introduced in Remark 3.8). Equivariance follows from Lemma A.5. □

Remark 4.8. If ρ is p-dominated then it satisfies the hypothesis of Lemma 4.7, since the terms in the sum of (4.3) are uniformly exponentially small.

Proposition 4.9. *Let $\rho : \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a p-dominated representation. Then ρ is p-Anosov.*

Proof. If a representation is p-dominated, then it is also $(d - p)$-dominated (see Remark 3.1). Lemma 4.7 then provides two equivariant continuous maps $\xi : \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\theta : \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$.

The fact that $\xi(x) \oplus \theta(y) = \mathbb{R}^d$ for $x \neq y \in \partial\Gamma$ is a direct consequence of Lemma 3.10 and the definition of the maps ξ and θ given by Lemma 4.7.

Using Lemma 4.4(ii), we obtain an exponential gap in the singular values of ψ^t, and by Theorem 2.2, the splitting $\xi \oplus \theta$ is dominated. Therefore ρ is p-Anosov. $\qquad\square$

4.4. Some questions

Given a matrix $A \in \mathrm{GL}(d, \mathbb{R})$, let

$$\chi_1(A) \geq \cdots \geq \chi_d(A)$$

denote the absolute values of its eigenvalues, repeated according to multiplicity.

Given a finitely generated group, let

$$\ell(\eta) := \inf_\eta |\eta^{-1}\gamma\eta| = \inf_\eta d(\gamma\eta, \eta)$$

(i.e., the *translation length*). If Γ is word-hyperbolic then there exists a constant $a > 0$ such that

$$\ell(\gamma) - a \leq \lim_{n\to\infty} \frac{|\gamma^n|}{n} \leq \ell(\gamma) \quad \text{for every } \gamma \in \Gamma \tag{4.4}$$

(see [CDP, p. 119]).

Note that if $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ is a p-dominated representation then there exist constants $C', \lambda > 0$ such that

$$\frac{\chi_{p+1}(\rho(\gamma))}{\chi_p(\rho(\gamma))} < C' e^{-\lambda'\ell(\gamma)} \quad \text{for all } \gamma \in \Gamma. \tag{4.5}$$

Indeed, if the domination condition (3.2) holds then the group Γ is word-hyperbolic by Theorem 3.2, and using (4.4) we obtain

$$\frac{\chi_{p+1}(\rho(\gamma))}{\chi_p(\rho(\gamma))} = \lim_{n\to\infty} \left(\frac{\sigma_{p+1}(\rho(\gamma^n))}{\sigma_p(\rho(\gamma^n))} \right)^{1/n} \leq \lim_{n\to\infty} (Ce^{-\lambda|\gamma^n|})^{1/n} \leq C' e^{-\lambda\ell(\gamma)}$$

for $C' := e^{a\lambda}$.

Condition (4.5) is invariant under conjugacies, while (3.2) is not.

It is natural to pose the following question:

Question 4.10. Let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation of a finitely generated group Γ. Suppose that there exist constants $p \in \{1, \ldots, d-1\}$ and $C', \lambda > 0$ such that (4.5) holds. Does it follow that ρ is p-dominated?

Guéritaud, Guichard, Kassel, and Wienhard [GGKW, Theorem 1.6] have shown that for p-convex representations, the question has a positive answer, even with (4.5) relaxed.[8]

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[8] Kassel has informed us that techniques similar to those in [GGKW] allow one to give a positive answer to Question 4.10 for certain word-hyperbolic groups, including free groups and surface groups.

In terms of the linear flow $\{\psi^t\}$, condition (4.5) means that for every periodic orbit $\mathcal{O}$ of $\{\phi^t\}$, say of period $\ell(\mathcal{O})$, there exists a gap between the p-th and $p+1$-th moduli of the eigenvalues of $\psi^{\ell(\mathcal{O})}$ which is exponentially large with respect to $\ell(\mathcal{O})$. Question 4.10 can be reformulated in the general context of linear flows over hyperbolic dynamics; however, that question has a negative answer: see for example [Go]. Therefore a positive answer to Question 4.10 would require a finer use of the fact that the linear flow comes from a representation.

The following important result was obtained by Bonatti, Díaz, and Pujals [BDP]: if a diffeomorphism f of a compact manifold has the property that all sufficiently small C^1-perturbations have dense orbits, then the derivative cocycle Df admits a dominated splitting. This is an example of a general principle in differentiable dynamics, going back to the Palis–Smale Stability Conjecture: robust dynamical properties often imply some uniform property for the derivative. Coming back to the context of linear representations, one can try to apply the same principle. For example, if a representation ρ of a hyperbolic group is robustly faithful and discrete (or robustly quasi-isometric), does it follow that ρ is p-dominated for some p?

5. Characterizing dominated representations in terms of multicones

The main result of this section is Theorem 5.9, which gives another characterization of dominated representations. Related results have been obtained in [ABY, BG]. In dimension two, such results have been used to study how domination can break along a deformation: see [ABY, §4].

As a consequence of Theorem 5.9, domination obeys a "local-to-global" principle, a fact that was first shown in [KLP$_1$] by different methods.

5.1. Sofic linear cocycles and a general multicone theorem

In this subsection we introduce a special class of linear cocycles called *sofic*. Then we state a necessary and sufficient condition for the existence of a dominated splitting for these cocycles, generalizing the "multicone theorems" of [ABY, BG].

Let $\mathcal{G}$ be a *graph*, or more precisely, a finite directed multigraph. This means that we are given finite sets $\mathcal{V}$ and $\mathcal{E}$ whose elements are called *vertices* and *edges* respectively, and that each edge has two (not necessarily different) associated vertices, called its *tail* and its *head*. A *bi-infinite walk* on $\mathcal{G}$ is a two-sided sequence $(e_n)_{n\in\mathbb{Z}}$ of edges such that for each $n \in \mathbb{Z}$, the head of e_n equals the tail of e_{n+1}.

The graph $\mathcal{G}$ is called *labeled* if in addition each edge has an associated *label*, taking values in some finite set $\mathcal{L}$. Let $(e_n)_{n\in\mathbb{Z}}$ be a bi-infinite walk on $\mathcal{G}$; then its *label sequence* is defined as $(\ell_n)_{n\in\mathbb{Z}}$ where each ℓ_n is the label of the edge e_n. Let Λ be the set of all label sequences; this is a closed, shift-invariant subset of $\mathcal{L}^\mathbb{Z}$. Let $T\colon \Lambda \to \Lambda$ denote the restriction of the shift map. Then T is called a *sofic shift*, and the labeled graph from which it originates is called a *presentation* of T. We refer the reader to [LM] for examples, properties, and alternative characterizations of sofic shifts. Let us just remark that every

subshift of finite type is a sofic shift, and every sofic shift is a factor of a subshift of finite type.

Let us say that a graph is *recurrent* if it is a union of directed cycles. Given a (labeled) graph $\mathcal{G}$, let $\mathcal{G}^*$ denote the maximal recurrent (labeled) subgraph, or equivalently the subgraph containing all the bi-infinite walks on $\mathcal{G}$. As the sofic shifts presented by $\mathcal{G}$ and $\mathcal{G}^*$ are exactly the same, we may always assume that $\mathcal{G}$ is recurrent if necessary.

Fix a sofic shift T and a presentation $\mathcal{G}$ as above. Let $d \geq 2$ be an integer. Given a family $(A_\ell)_{\ell \in \mathcal{L}}$ of matrices in $\mathrm{GL}(d, \mathbb{R})$, consider the map $A : \Lambda \to \mathrm{GL}(d, \mathbb{R})$ defined by $A((\ell_n)_{n \in \mathbb{Z}}) := A_{\ell_0}$. We call the pair (T, A) a *sofic linear cocycle*. We are interested in the existence of dominated splitting for such cocycles.

A *multicone of index p* is an open subset of the projective space $\mathbb{P}(\mathbb{R}^d) = \mathscr{G}_1(\mathbb{R}^d)$ that contains the projectivization of some p-plane and does not intersect the projectivization of some $(d - p)$-plane. Such a multicone is called *tame* if it has finitely many connected components, and the components have disjoint closures.

Suppose that for each vertex v of the graph $\mathcal{G}$ there is given a multicone $M_v \subset \mathbb{P}(\mathbb{R}^d)$ of index p; then we say that $(M_v)_{v \in \mathcal{V}}$ is a *family of multicones of index p*. We say that this family is *strictly invariant* (with respect to the sofic linear cocycle) if[9]

$$A_\ell(M_{v_0}) \Subset M_{v_1} \quad \text{for each edge } e \in \mathcal{E},$$

where ℓ is the label of e, v_0 is the tail of e, and v_1 is the head of e.

Theorem 5.1. *Let T be a sofic shift with a fixed presentation. Consider a sofic linear cocycle (T, A). The following statements are equivalent:*

(a) *the cocycle (T, A) has a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ where the dominating bundle E^{cu} has dimension p;*

(b) *there exists a strictly invariant family of multicones of index p.*

Moreover, in (b) *we can always choose a family composed of tame multicones.*

Remark 5.2. It is not always possible to choose *connected* multicones in (b). Let us sketch the simplest example, referring the reader to [ABY] for more information. Let T be the full shift on two symbols 1, 2, presented by the graph $1 \circlearrowright \bullet \circlearrowleft 2$. Let A_1, A_2 be matrices in $\mathrm{SL}(2, \mathbb{R})$ whose unstable and stable directions in $\mathbb{P}(\mathbb{R}^2)$ can be cyclically ordered as

$$E^{\mathrm{u}}(A_1) < E^{\mathrm{s}}(A_1) < E^{\mathrm{u}}(A_2) < E^{\mathrm{s}}(A_2).$$

If the eigenvalues are sufficiently away from 1 then the sofic cocycle (T, A) has a dominated splitting. A strictly invariant family of multicones consists of a single element, since the graph has a single vertex. It is possible to chose a multicone $M \subset \mathbb{P}(\mathbb{R}^2)$ with two connected components. But M cannot be chosen connected, since it must contain $\{E^{\mathrm{u}}(A_1), E^{\mathrm{u}}(A_2)\}$ and it cannot intersect $\{E^{\mathrm{s}}(A_1), E^{\mathrm{s}}(A_2)\}$.

---

[9] $X \Subset Y$ denotes that the closure of X is contained in the interior of Y.

Theorem 5.1 implies, for example, [ABY, Theorem 2.2]; to see this, note that the full shift on N symbols can be presented by the graph with a single vertex and self-loops labeled $1, \ldots, N$. Theorem 5.1 also extends [ABY, Theorem 2.3] and [BG, Theorem B] (in the case of finite sets of matrices).

One can prove Theorem 5.1 by a minor adaptation of the proof of [BG, Theorem B]; for the convenience of the reader we provide details in Subsection 5.2.

As a complement to Theorem 5.1, let us explain how to obtain the dominating bundle E^{cu} in terms of the multicones:

Proposition 5.3. *Let* (T, A) *be a sofic linear cocycle with a dominated splitting* $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ *where the dominating bundle* E^{cu} *has dimension* p. *Let* $\{M_v\}_v$ *be a strictly invariant family of multicones. Let* $x = (\ell_n)_{n \in \mathbb{Z}} \in \Lambda$, *the label sequence of a bi-infinite walk* $(e_n)_{n \in \mathbb{Z}}$. *Let* v_n *be the tail of the edge* e_n. *Let* $(P_n)_{n \in \mathbb{Z}}$ *be a sequence in* $\mathscr{G}_p(\mathbb{R}^d)$ *such that for each* n, *the projectivization of* P_n *is contained in the multicone* M_{v_n}. *Then*

$$E^{\mathrm{cu}}(x) = \lim_{n \to +\infty} A_{\ell_{-1}} A_{\ell_{-2}} \cdots A_{\ell_{-n}} (P_{-n}).$$

Moreover, the speed of convergence is exponential and can be estimated independently of x, (e_n), *and* (P_n), *and is the same for all nearby sofic linear cocycles* $(T, \tilde{A})$.

5.2. How to prove Theorem 5.1 and Proposition 5.3

The implication (b)⇒(a) in Theorem 5.1 and Proposition 5.3 actually hold in much greater generality:

Proposition 5.4. *If a linear cocycle*[10] (T, A) *admits a strictly invariant continuous field* $\mathcal{C}$ *of multicones of index* p, *then the cocycle admits a dominated splitting* $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ *where the dominating bundle* E^{cu} *has dimension* p. *Moreover,*

$$E^{\mathrm{cu}}(x) \in \mathcal{C}(x) \quad and \quad \inf_{P \in \mathcal{C}(x)} \angle(P, E^{\mathrm{cs}}(x)) > 0 \quad for\ every\ x. \tag{5.1}$$

The proposition can be shown by a straightforward adaptation of the proof of [CP, Theorem 2.6], and we skip the details.

Proposition 5.3 follows from the angle bound in (5.1) combined with Lemma A.10.

We now prove the implication (a)⇒(b) in Theorem 5.1. Suppose that the sofic cocycle (T, A) admits a dominated splitting $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$. For each vertex v of the graph $\mathcal{G}$, let Λ_v be the (compact) set of label sequences $x = (\ell_n)$ in Λ for which the tail of ℓ_0 is v. The space $E^{\mathrm{cu}}(x)$ depends only on the past of the sequence (i.e. on $(\ell_n)_{n<0}$) while $E^{\mathrm{cs}}(x)$ depends only on its future (i.e. on $(\ell_n)_{n \geq 0}$); this follows from the characterizations (2.3) and (2.4).

For each vertex v, the set $\mathcal{K}_v^{\mathrm{cu}}$ consisting of the spaces $E^{\mathrm{cu}}(x)$ with $x \in \Lambda_v$ is a compact subset of the Grassmannian $\mathscr{G}_p(\mathbb{R}^d)$. Analogously we define a compact set $\mathcal{K}_v^{\mathrm{cs}} \subset \mathscr{G}_{d-p}(\mathbb{R}^d)$. We claim that these two sets are *transverse* in the sense that any $E^{\mathrm{cu}}(x)$

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[10] Or, more generally, a linear flow in the context of Subsection 2.1

in $\mathcal{K}_v^{\mathrm{cu}}$ is transverse to any $E^{\mathrm{cu}}(y)$ in $\mathcal{K}_v^{\mathrm{cs}}$. Indeed, consider the sequence z formed by concatenating the past of x with the future of y. Note that this is indeed the label sequence of a bi-infinite path (i.e. an admissible sequence of edges) in the graph, also obtained by concatenation. By a previous remark, $E^{\mathrm{cu}}(z) = E^{\mathrm{cu}}(x)$ and $E^{\mathrm{cu}}(z) = E^{\mathrm{cu}}(y)$, and so these two spaces are transverse.

Let $\mathcal{K}_v^{\pitchfork\mathrm{cs}}$ be the open subset of $\mathscr{G}_p(\mathbb{R}^d)$ formed by the p-planes that are transverse to $\mathcal{K}_v^{\mathrm{cs}}$. Define a metric d_v on $\mathcal{K}_v^{\pitchfork\mathrm{cs}}$ as follows:

$$d_v(P, Q) := \sum_{N=0}^{\infty} \sup_{(\ell_n) \in \Lambda_v} d\big(A_{\ell_{N-1}} \cdots A_{\ell_0}(P), A_{\ell_{N-1}} \cdots A_{\ell_0}(Q)\big);$$

the convergence is exponential as a consequence of Lemma A.10. This family of metrics is *adapted* in the sense that they are contracted by a single iteration of the cocycle. More precisely, if an edge $e \in \mathcal{E}$ has tail v, head w, and label ℓ then

$$d_w(A_\ell(P), A_\ell(Q)) < d_v(P, Q) \quad \text{for all } P, Q \in \mathcal{K}_v^{\pitchfork\mathrm{cs}}.$$

So, fixing $\varepsilon > 0$ and letting M_v^* denote the ε-neighborhood of $\mathcal{K}_v^{\mathrm{cu}}$ with respect to the metric d_v, we have the invariance property $A_\ell(M_v^*) \subset M_w^*$. Let $M_v \subset \mathbb{P}(\mathbb{R}^d)$ be the union of all lines contained in elements of M_v^*; then $\{M_v\}_v$ is an invariant family of multicones. Moreover, by the same argument as in [BG, p. 288], for an appropriate choice of ε the multicones are tame and strictly invariant.

5.3. Cone types for word-hyperbolic groups

Cone types were originally introduced by Cannon [Ca] for groups of hyperbolic isometries.

Let Γ be a finitely generated group with a fixed finite symmetric generating set S. Let $|\cdot|$ and $d(\cdot, \cdot)$ denote the word-length and the word-metric respectively.

The *cone type* of an element $\gamma \in \Gamma$ is defined as

$$C^+(\gamma) := \{\eta \in \Gamma : |\eta\gamma| = |\eta| + |\gamma|\}.$$

For example, $C^+(\gamma) = \Gamma$ if and only if $\gamma = \mathrm{id}$.

Remark 5.5. Actually the usual definition is different:

$$C^-(\gamma) := \{\eta \in \Gamma : |\gamma\eta| = |\eta| + |\gamma|\}.$$

But working with one definition is essentially equivalent to working with the other, because $C^-(\gamma) = [C^+(\gamma^{-1})]^{-1}$. If we need to distinguish between the two, we shall call them *positive* and *negative* cone types.

A fundamental fact is that word-hyperbolic groups have only finitely many cone types: see [BH, p. 455] or [CDP, p. 145]. In fact, there is a constant k (depending only on the hyperbolicity constant of the group) such that for any $\gamma \in \Gamma$, the cone type $C^+(\gamma)$ is uniquely determined by the k-prefix[11] of a shortest word representation of γ.

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[11] Or k-suffix in the case of negative cone types.

Given a cone type C and $a \in S \cap C$, we can define a cone type

$$aC := C^+(a\gamma),$$

where $\gamma \in \Gamma$ is such that $C^+(\gamma) = C$.

Lemma 5.6. *aC is well-defined.*

Though the lemma is contained in [CDP, p. 147, Lemme 4.3], let us provide a proof for the reader's convenience:

Proof of Lemma. Suppose that $C^+(\gamma) = C^+(\gamma') = C$ and $a \in S \cap C$; we need to prove that $C^+(a\gamma) \subset C^+(a\gamma')$. Take $\eta \in C^+(a\gamma)$, so $|\eta a \gamma| = |\eta| + |a\gamma| = |\eta| + 1 + |\gamma|$. In follows that $|\eta a| = |\eta| + 1$ and so $|\eta a \gamma| = |\eta a| + |\gamma|$, that is, $\eta a \in C$. In particular $|\eta a \gamma'| = |\eta a| + |\gamma'| = |\eta| + |a\gamma|$, proving that $\eta \in C^+(a\gamma')$. $\square$

We associate to (Γ, S) a labeled graph $\mathcal{G}$ (in the sense of §5.1) called the *geodesic automaton* and defined as follows:

- the vertices are the cone types of Γ;
- there is an edge $C_1 \xrightarrow{a} C_2$ from vertex C_1 to vertex C_2, labeled by a generator $a \in S$, iff $a \in C_1$ and $C_2 = aC_1$.

Remark 5.7. Replacing each vertex C by C^{-1} (a negative cone type) and each edge $C_1 \xrightarrow{a} C_2$ by $C_1^{-1} \xrightarrow{a^{-1}} C_2^{-1}$, we obtain the graph described in [BH, p. 456].

Let us explain the relation to geodesics. Consider a *geodesic segment* $(\gamma_0, \gamma_1, \ldots, \gamma_\ell)$ that is, a sequence of elements of Γ such that $d(\gamma_n, \gamma_m) = |n - m|$, and assume that $\gamma_0 = \mathrm{id}$. Then there are $a_0, \ldots, a_{\ell-1}$ in the generating set S such that $\gamma_n = a_0 a_1 \cdots a_{n-1}$. Note that for each n, the following is an edge of the geodesic automaton graph $\mathcal{G}$:

$$C^+(\gamma_n^{-1}) \xrightarrow{a_n^{-1}} C^+(\gamma_{n+1}^{-1}).$$

Thus we obtain a (finite) walk on $\mathcal{G}$ starting from the vertex $C^+(\mathrm{id})$. Conversely, to each such walk we may associate a geodesic segment starting at the identity.

Let us also define the *recurrent geodesic automaton* as the maximal recurrent subgraph $\mathcal{G}^*$ of $\mathcal{G}$; its vertices are called *recurrent cone types*. Similarly to what was explained above, to each two-sided geodesic on Γ passing through the identity element we can associate a bi-infinite walk on $\mathcal{G}^*$, and vice versa.

Using the fact that the geodesic automaton is a finite graph, it is easy to obtain the following property:[12]

Lemma 5.8. *Let Γ be an infinite word-hyperbolic group, with a fixed set of generators. Let Γ^* be the union of all two-sided geodesics passing through the identity element. Then for some finite c, the set Γ^* is c-dense on Γ, that is, for every $\gamma \in \Gamma$ there exists $\eta \in \Gamma^*$ such that $d(\eta, \gamma) \leq c$.*

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[12] Alternatively, one can deduce the lemma from the "bounded dead-end depth" property [Bog].

5.4. Multicones for dominated representations

Let Γ be a word-hyperbolic group (with a fixed finite symmetric generating set), and let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation.

Recall from §5.1 that a multicone of index p is an open subset of $\mathbb{P}(\mathbb{R}^d)$ that contains the projectivization of some p-plane and does not intersect the projectivization of some $(d - p)$-plane.

If for each recurrent cone type C, a multicone $M(C)$ of index p is given, then we say that $\{M(C)\}_C$ is a *family of multicones* for Γ. We say that this family is *strictly invariant* with respect to ρ if for each edge $C_1 \xrightarrow{g} C_2$ of the geodesic automaton graph, we have

$$\rho(g)(M(C_1)) \Subset M(C_2).$$

Theorem 5.9. *A representation of a word-hyperbolic group is p-dominated if and only if it has a strictly invariant family of multicones of index p. Moreover, we can always choose a family composed of tame multicones.*

Proof. Fix a word-hyperbolic group Γ. We can assume that Γ is infinite, otherwise the theorem is vacuously true. Fix a finite symmetric generating set S. Consider the associated recurrent geodesic automaton $\mathcal{G}^*$ and the sofic shift $T \colon \Lambda \to \Lambda$ presented by this labeled graph. Then a sequence (a_n) in $S^{\mathbb{Z}}$ belongs to Λ if and only if the sequence $(\gamma_n)_{n \in \mathbb{Z}}$ defined by

$$\gamma_n := \begin{cases} a_0^{-1} a_1^{-1} \cdots a_{n-1}^{-1} & \text{if } n > 0, \\ \mathrm{id} & \text{if } n = 0, \\ a_{-1} a_{-2} \cdots a_n & \text{if } n < 0, \end{cases}$$

is a geodesic on Γ. The union of (the traces of) all such geodesics is a set $\Gamma^* \subset \Gamma$ which by Lemma 5.8 is c-dense in Γ for some finite c.

Let $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$ be a representation. Consider the family $(\rho(a))_{a \in S}$ of matrices, and let (T, A) be the induced sofic linear cocycle. Then, for each $x = (a_n) \in X$, if (γ_n) is the geodesic defined above then $A(T^{n-1}x) \cdots A(x) = \rho(\gamma_n^{-1})$ for every positive n.

By Remark 3.1, ρ is p-dominated iff it is $(d - p)$-dominated. Since the set Γ^* is c-dense, ρ is $(d - p)$-dominated iff

$$\exists C > 0 \; \exists \lambda > 0 \; \forall \gamma \in \Gamma^*, \quad \frac{\sigma_{p+1}}{\sigma_p}(\rho(\gamma^{-1})) \leq C e^{-\lambda |\gamma|}.$$

Note that this condition holds iff the sofic linear cocycle (T, A) has a dominated splitting with a dominating bundle of dimension p; this follows from Theorem 2.2 and the previous observations. Also note that a strictly invariant family of multicones for the sofic linear cocycle (T, A) is exactly the same as a strictly invariant family of multicones for the representation ρ. Therefore Theorem 5.1 allows us to conclude. $\quad\square$

Theorem 5.9 also has the following well-known consequence (see [GW, Lab]).

Corollary 5.10. *Among representations $\Gamma \to \mathrm{GL}(d, \mathbb{R})$, the p-dominated ones form an open set.*

Proof. Given a p-dominated representation, take a strictly invariant family of multicones. Then the same family is also strictly invariant under all nearby representations. $\quad\square$

As a complement to Theorem 5.9, let us explain how to determine the equivariant map $\xi \colon \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ (defined in Section 4) in terms of multicones.

Proposition 5.11. *Consider a p-dominated representation $\rho \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})$, and let $\{M(C)\}_C$ be a strictly invariant family of multicones. Consider any geodesic ray $(\gamma_n)_{n \in \mathbb{N}}$ with $\gamma_0 = \mathrm{id}$, and let $x \in \partial\Gamma$ be the associated boundary point. Let $(P_n)_{n \geq 1}$ be a sequence in $\mathscr{G}_p(\mathbb{R}^d)$ such that for each n, the projectivization of P_n is contained in the multicone $M(C^+(\gamma_n))$. Then*

$$\xi(x) = \lim_{n \to \infty} \rho(\gamma_n^{-1}) P_n. \tag{5.2}$$

Moreover, the speed of convergence can be estimated independently of x, (γ_n), and (P_n), and is the same for all nearby representations.

Proof. It suffices to translate Proposition 5.3 to the context of representations. $\square$

Remark 5.12. Recall from Remark 3.4 that it also makes sense to define p-dominated representations in $\mathrm{PGL}(d, \mathbb{R})$. All that was said in this subsection applies verbatim to that case, since $\mathrm{PGL}(d, \mathbb{R})$ also acts on $\mathbb{P}(\mathbb{R}^d)$.

5.5. An example

Consider the free product $\Gamma := (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ (which is isomorphic to $\mathrm{PGL}(2, \mathbb{Z})$) with a presentation

$$\Gamma = \langle a, b \mid a^3 = b^2 = \mathrm{id} \rangle.$$

Then there are only two recurrent cone types, namely $C^+(a)$ and $C^+(b)$, and the recurrent geodesic automaton is
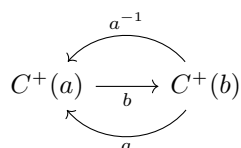
Fix $\lambda > 1$. Consider the following pair of matrices in $\mathrm{SL}(2, \mathbb{R})$:

$$A := D^{-1} R_{\pi/3} D \quad \text{and} \quad B := R_{\pi/2},$$

where

$$D := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \quad \text{and} \quad R_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Since $A^3 = B^2 = -\mathrm{id}$, we can define a representation $\rho \colon \Gamma \to \mathrm{PSL}(2, \mathbb{R})$ by setting $\rho(a) := A$ and $\rho(b) := B$. We claim that if λ is sufficiently large (namely, $\lambda > \sqrt[4]{2}$) then this representation is dominated. Indeed, it is possible to find a strictly invariant family of multicones as in Fig. 5.
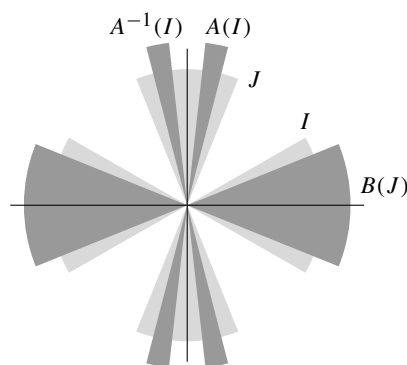
Fig. 5. A strictly invariant multicone for the representation $\rho \colon (\mathbb{Z}/3\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z}) \to \mathrm{PSL}(2, \mathbb{R})$. We have $A(I) \Subset J$, $A^{-1}(I) \Subset J$, $B(J) \Subset I$.

6. Analytic variation of limit maps

The purpose of this section is to give another proof of a theorem from [BCLS], which establishes that the equivariant limit maps ξ, η (defined in Section 4) depend analytically on the representation. This fact is useful to show that some quantities such as entropy vary analytically with respect to the representation, which in turn is important to obtain certain rigidity results (see [BCLS, PS]).

While the original approach of [BCLS] used the formalism of [HPS], we present here a more direct proof, based on the tools discussed in the previous section. We remark that an alternative approach using the implicit function theorem and avoiding complexification can be found in [Ru], though the context is different and the results are not exactly the same.

A family $\{\rho_u\}_{u \in D}$ of representations of a word-hyperbolic group Γ in $\mathrm{GL}(d, \mathbb{R})$ is called *real analytic* if the parameter space is a real analytic manifold, and for each $\gamma \in \Gamma$, the map $D \ni u \mapsto \rho_u(\gamma) \in \mathrm{GL}(d, \mathbb{R})$ is real analytic. (Of course, it suffices to check the latter condition for a set of generators.)

The boundary $\partial\Gamma$ of the group Γ admits a distance function within a "canonical" Hölder class [CDP, Chapitre 11]. For a fixed metric in this class, the geodesic flow defined previously is Lipschitz. The limit maps $\xi_\rho \colon \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ and $\theta_\rho \colon \partial\Gamma \to \mathscr{G}_{d-p}(\mathbb{R}^d)$ of a p-dominated representation are well-known to be Hölder continuous (see Theorem A.15). One can endow the space $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$ of α-Hölder maps with a Banach manifold structure, so analyticity of a map from an analytic manifold to $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$ makes sense.

Theorem 6.1 ([BCLS, Theorem 6.1]). *Let $\{\rho_u \colon \Gamma \to \mathrm{GL}(d, \mathbb{R})\}_{u \in D}$ be a real analytic family of representations. Suppose that $0 \in D$ and ρ_0 is p-dominated. Then there exists a neighborhood $D' \subset D$ of 0 such that for every $u \in D'$, the representation ρ_u is p-dominated, and moreover $u \mapsto \xi_{\rho_u}$ defines a real analytic map from D' to $C^\alpha(\partial\Gamma, \mathscr{G}_p(\mathbb{R}^d))$ for some $\alpha > 0$.*

Let us provide a proof. Corollary 5.10 ensures that for every u sufficiently close to 0, the representation ρ_u is also p-dominated.

As in [BCLS] (see also [Hub, Proposition A.5.9]), it is enough to show *transverse real analyticity*. More precisely, we need to show that the map $F: D' \times \partial\Gamma \to \mathscr{G}_p(\mathbb{R}^d)$ given by $F(u, x) := \xi_{\rho_u}(x)$ has the following properties:

(i) it is α-Hölder continuous;
(ii) for every $x \in \partial\Gamma$, the map $F(\cdot, x): D' \to \mathscr{G}_p(\mathbb{R}^d)$ is real analytic.

Hölder continuity is a standard property of dominated splittings (see for example [CP, Section 4.4]), and is independent of the analyticity of the family. For completeness, and since we could not find the specific statement in the literature, we include a sketch of the proof in the appendix: see Corollary A.16. In the case at hand, it follows that property (i) above is satisfied, for some neighborhood $D' \ni 0$ and some uniform Hölder exponent $\alpha > 0$.

So it remains to prove the analyticity property (ii). To proceed further, we consider the complexification of the representations. We can assume without loss of generality that D' is a neighborhood of 0 in some $\mathbb{R}^k$. Let $\{g_1, \ldots, g_m\}$ be a finite generating set of Γ. For each i, the Taylor expansion of $u \mapsto \rho_u(g_i)$ around 0 converges on a polydisk $\hat{D}_i$ in $\mathbb{C}^k$ centered at 0; we keep the same symbol for the extended map. Take a smaller polydisk $D \subset \bigcap_i \hat{D}_i$ also centered at 0 such that for $u \in D$, the complex matrix $\rho_u(g_i)$ is invertible. Since these maps are analytic, and every relation of Γ is obeyed when u is real, the same happens for all $u \in \hat{D}$. So we have constructed a complex analytic family of representations $\rho_u: \Gamma \to \mathrm{GL}(d, \mathbb{C})$, where u takes values in a neighborhood $\hat{D}$ of 0 in $\mathbb{C}^k$.

Recall from Remark 2.3 that dominated splittings make sense in the complex case; so do dominated representations, with the exact same definition (3.2). Actually, if ι denotes the usual homomorphism that embeds $\mathrm{GL}(d, \mathbb{C})$ into $\mathrm{GL}(2d, \mathbb{R})$, then a representation $\rho: \Gamma \to \mathrm{GL}(d, \mathbb{C})$ is p-dominated in the complex sense iff $\iota \circ \rho$ is $2p$-dominated in the real sense. In particular, the openness property of Corollary 5.10 also holds in the complex case. So, reducing $\hat{D}$ if necessary, we assume that each ρ_u is p-dominated.

Proposition 6.2. *For each $x \in \partial\Gamma$, the map $u \mapsto \xi_u(x)$ from $\hat{D}$ to $\mathscr{G}_p(\mathbb{C}^d)$ is complex analytic.*

Proof. It suffices to check analyticity around $u = 0$. For each $x \in \partial\Gamma$, choose a geodesic ray $(\gamma_n^x)_{n \in \mathbb{N}}$ in Γ starting at the identity element and converging to x. We consider the sequence of maps from $\hat{D}$ to $\mathscr{G}_p(\mathbb{C}^d)$ defined as follows:

$$\varphi_n^x(u) := \rho_u(\gamma_n^x)^{-1}(\xi_0(\gamma_n^x(x))).$$

Each of these maps is complex analytic, since so is $u \mapsto \rho_u(\gamma_n^x)$.

We claim that for each $x \in \partial\Gamma$, the convergence $\varphi_n^x(u) \to \xi_u(x)$ holds uniformly in a neighborhood of 0. Since $\mathscr{G}_p(\mathbb{C}^d)$ may be considered as a closed subset of $\mathscr{G}_{2p}(\mathbb{R}^{2d})$, it is sufficient to check convergence in the latter set. This convergence follows from Proposition 5.11 and the fact that for u in a small neighborhood V of 0, the representations are

still dominated, and with the same multicones (see Corollary 5.10). Note that ξ_0 always belongs to the corresponding multicone because the multicones remain unchanged.

Being a uniform limit of complex analytic maps, the map $u \mapsto \xi_u(x)$ is complex analytic on V (see for example [Hö, Corollary 2.2.4]). □

Restricting to the real parameters, the proposition yields the property (ii) that we were to check. So the proof of Theorem 6.1 is complete.

7. Geometric consequences of Theorem 2.2: A Morse Lemma for $PSL(d, \mathbb{R})$'s symmetric space

In this section we explain why Theorem 2.2 (and more precisely Proposition 2.4) has a deep geometric meaning for the symmetric space of $PSL(d, \mathbb{R})$. This is a version of the Morse Lemma recently proved by [KLP₂]. Because of this application, one is tempted to call Theorem 2.2 a *twisted Morse Lemma*.[13]

The exposition is purposely pedestrian for two reasons: (1) it is intended for the reader unfamiliar with symmetric spaces; (2) it mimics, in the case of $PSL(d, \mathbb{R})$, the general structure theory of semisimple Lie groups, in order to ease the way to Section 8.

The reader familiar with semisimple Lie groups should jump to Subsection 7.12, or even Section 8, for a proof of the Morse Lemma due to [KLP₂], for symmetric spaces of noncompact type, using dominated splittings. Specific references for Subsections 7.1–7.11 are, for example, [Ebe], [GJT], [Hel], [Lang].

This section is independent of Sections 3–6.

7.1. A Cartan subalgebra

Fix an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ and denote by o its homothety class, i.e. $\langle \cdot, \cdot \rangle$ up to positive scalars. One then has the adjoint involution $T \mapsto T^{\mathsf{t}}$ defined by $\langle Tv, w \rangle = \langle v, T^{\mathsf{t}}w \rangle$ (note that $^{\mathsf{t}}$ only depends on o). This involution splits the vector space $\mathfrak{sl}(d, \mathbb{R})$ of traceless $d \times d$ matrices as

$$\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{p}^o \oplus \mathfrak{k}^o$$

where

$$\mathfrak{p}^o = \{T \in \mathfrak{sl}(d, \mathbb{R}) : T^{\mathsf{t}} = T\} \quad \text{and} \quad \mathfrak{k}^o = \{T \in \mathfrak{sl}(d, \mathbb{R}) : T^{\mathsf{t}} = -T\}.$$

The subspace $\mathfrak{k}^o$ is a Lie algebra so we can consider its associated Lie group $K^o = \exp \mathfrak{k}^o$, consisting of (the projectivizations of) determinant 1 matrices preserving the class o. The subspace $\mathfrak{p}^o$, of traceless matrices diagonalizable for an o-orthogonal basis, is not a Lie algebra.

Fix then an o-orthogonal set of d lines $\mathcal{E}$ and denote by $\mathfrak{a} \subset \mathfrak{p}^o$ those matrices diagonalizable in the chosen set $\mathcal{E}$. This is an abelian algebra, called a *Cartan subalgebra*

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[13] Lenz [Len], who previously obtained a weaker version of Theorem 2.2 for $SL(2, \mathbb{R})$, had already noted that his result was related to the classical Morse Lemma for the hyperbolic plane.

of $\mathfrak{sl}(d, \mathbb{R})$; its associated Lie group $\exp \mathfrak{a}$ consists of (the projectivizations of) determinant 1 matrices diagonalizable on $\mathcal{E}$ with positive eigenvalues. For $a \in \mathfrak{a}$ and $u \in \mathcal{E}$, we will denote by $\lambda_u(a)$ the eigenvalue for a associated to the eigenline u. Note that λ_u is linear in a and hence an element of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$.

7.2. The action of $\mathfrak{a}$ on $\mathfrak{sl}(d, \mathbb{R})$

The action of $\mathfrak{a}$ on $\mathfrak{sl}(d, \mathbb{R})$ given by $(a, T) \mapsto [a, T] = aT - Ta$ is also diagonalizable. Indeed, the set of (projective) traceless transformations $\{\varepsilon_{uv}, \phi_{uv} : u \neq v \in \mathcal{E}\}$ defined by

$$\varepsilon_{uv}(v) = u, \qquad \varepsilon_{uv}|\mathcal{E} - \{v\} = 0,$$

and

$$\phi_{uv}|u = t \,\text{id}, \qquad \phi_{uv}|v = -t \,\text{id}, \qquad \phi_{uv}|\mathcal{E} - \{u, v\} = 0,$$

contains a linearly independent set[14] of eigenlines of $[a, \cdot]$, specifically $[a, \phi_{uv}] = 0$ and

$$[a, \varepsilon_{uv}] = \alpha_{uv}(a)\varepsilon_{uv} = (\lambda_u(a) - \lambda_v(a))\varepsilon_{uv}.$$

The set of functionals

$$\Sigma = \{\alpha_{uv} \in \mathfrak{a}^* : u \neq v \in \mathcal{E}\}$$

is called a *root system* (or simply the *roots*) of $\mathfrak{a}$. For $\alpha \in \Sigma \cup \{0\}$, one usually denotes by $\mathfrak{sl}(d, \mathbb{R})_\alpha$ the eigenspace associated to α, that is,

$$\mathfrak{sl}(d, \mathbb{R})_\alpha = \{T \in \mathfrak{sl}(d, \mathbb{R}) : [a, T] = \alpha(a)T, \, \forall a \in \mathfrak{a}\},$$

and one has[15]

$$\mathfrak{sl}(d, \mathbb{R}) = \bigoplus_{\alpha \in \Sigma \cup \{0\}} \mathfrak{sl}(d, \mathbb{R})_\alpha = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma} \mathfrak{sl}(d, \mathbb{R})_\alpha. \tag{7.1}$$

7.3. Expansion/contraction

The closure of a connected component of

$$\mathfrak{a} - \bigcup_{\alpha \in \Sigma} \ker \alpha$$

is called a *closed Weyl chamber*. Fix a closed Weyl chamber and denote it by $\mathfrak{a}^+$; this is not a canonical choice: it is equivalent to choosing an order on the set $\mathcal{E}$. Indeed, consider the subset of *positive roots* defined by $\mathfrak{a}^+$:

$$\Sigma^+ = \{\alpha \in \Sigma : \alpha|\mathfrak{a}^+ \geq 0\};$$

then one can set $u > v$ if $\alpha_{uv} \in \Sigma^+$.[16]

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[14] Redundancy only appears in the set $\{\phi_{uv} : u \neq v \in \mathcal{E}\}$.

[15] The fact that $\mathfrak{sl}(d, \mathbb{R})_0 = \mathfrak{a}$ is particular of split real algebras, such as $\mathfrak{sl}(d, \mathbb{R})$.

[16] In the case of $\mathrm{PSL}(d, \mathbb{R})$ one usually applies the inverse procedure: let $\{e_1, \ldots, e_d\}$ be the canonical basis of $\mathbb{R}^d$ and let $\mathfrak{a}^+$ be the set of (determinant one) diagonal matrices with decreasing eigenvalues.

Let $\mathfrak{n}^+$ be the Lie algebra defined by

$$\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{sl}(d, \mathbb{R})_\alpha = \bigoplus_{u > v} \varepsilon_{uv}.$$

By definition, $\mathfrak{a} \oplus \mathfrak{n}^+$ is the subspace of $\mathfrak{sl}(d, \mathbb{R})$ which is nonexpanded by $\operatorname{Ad} \exp a$ for $a \in \mathfrak{a}^+$.

7.4. Simple roots

Observe that the order on $\mathcal{E}$ defined in Subsection 7.3 is a total order; indeed, the kernel of every $\alpha \in \Sigma$ has empty intersection with the interior of $\mathfrak{a}^+$, so given distinct $u, v \in \mathcal{E}$, either $\alpha_{uv} \in \Sigma^+$ or $\alpha_{vu} = -\alpha_{uv} \in \Sigma^+$.

Consider pairwise distinct $u, v, w \in \mathcal{E}$ such that $u > v > w$. Note that if $a \in \mathfrak{a}^+ \cap \ker \alpha_{uw}$ then necessarily

$$a \in \ker \alpha_{uv} \cap \ker \alpha_{vw}.$$

A positive root α such that $\ker \alpha \cap \mathfrak{a}^+$ has maximal codimension is called a *simple root* associated to $\mathfrak{a}^+$. This corresponds to choosing two successive elements of $\mathcal{E}$. The set of simple roots is denoted Π. Note that this is a basis of $\mathfrak{a}^*$.

From now on we will denote $\mathcal{E} = \{u_1, \ldots, u_d\}$ with $u_p > u_{p+1}$. Then $\mathfrak{n}^+$ can be interpreted as the space of upper triangular matrices on $\mathcal{E}$ (with 0's on the diagonal), and denoting $a_i = \lambda_{u_i}(a)$ one has

$$\mathfrak{a}^+ = \{a \in \mathfrak{a} : a_1 \geq \cdots \geq a_d\}.$$

We will use λ_{u_i} to introduce coordinates in $\mathfrak{a}$: if $(a_1, \ldots, a_d) \in \mathbb{R}^d$ are such that $a_1 + \cdots + a_d = 0$ then $(a_1, \ldots, a_d)$ will denote the element $a \in \mathfrak{a}$ such that $\lambda_{u_i}(a) = a_i$. Finally, given $p \in \{1, \ldots, d-1\}$ we will denote by α_p the simple root

$$\alpha_p(a) = \alpha_{u_p u_{p+1}}(a) = a_p - a_{p+1}.$$

7.5. Flags

Denote $N = \exp \mathfrak{n}^+$ and let M be the centralizer of $\exp \mathfrak{a}$ in K^o. The group M consists of (the projectivizations of) diagonal matrices with respect to $\mathcal{E}$ with eigenvalues 1 and -1.

The group $P = M \exp \mathfrak{a} N$ is called a *Borel subgroup* of $\operatorname{PSL}(d, \mathbb{R})$ and N is called its *unipotent radical*.

Recall that a *complete flag* on $\mathbb{R}^d$ is a collection $E = \{E_p\}_{p=1}^{d-1}$ of subspaces such that $E_p \subset E_{p+1}$ and $\dim E_p = p$. The space of complete flags is denoted by $\mathscr{F}$. Observe that $\operatorname{PSL}(d, \mathbb{R})$ acts transitively on $\mathscr{F}$ and the group P is the stabilizer of

$$\{u_1 \oplus \cdots \oplus u_p\}_{p=1}^{d-1};$$

thus we obtain an equivariant identification $\mathscr{F} = \operatorname{PSL}(d, \mathbb{R})/P$.

Two complete flags E and F are *in general position* if

$$E_p \cap F_{d-p} = \{0\} \quad \text{for all } p = 1, \ldots, d-1.$$

Denote by $\mathscr{F}^{(2)}$ the space of pairs of flags in general position.

The same procedure applied to the Weyl chamber $-\mathfrak{a}^+$ provides the group $\check{P}$ that stabilizes the complete flag

$$\{u_d \oplus \cdots \oplus u_{d-p+1}\}_{p=1}^{d-1}.$$

Observe that the flags $\{u_1 \oplus \cdots \oplus u_p\}_{p=1}^{d-1}$ and $\{u_d \oplus \cdots \oplus u_{d-p+1}\}_{p=1}^{d-1}$ are in general position and the stabilizer in $\mathrm{PSL}(d, \mathbb{R})$ of the pair is the group $M \exp \mathfrak{a} = P \cap \check{P}$.

7.6. Flags and singular value decomposition

If $\langle \cdot, \cdot \rangle \in o$ is an inner product with induced norm $\|\cdot\|$ on $\mathbb{R}^d$, note that the operator norm of $g \in \mathrm{GL}(d, \mathbb{R})$,

$$\sigma_1^o(g) = \|g\|_o = \sup\{\|gv\|/\|v\| : v \in \mathbb{R}^d - \{0\}\},$$

only depends on the homothety class o. The same holds for the other singular values, defined in Subsection 2.2. Throughout this section, the choice of the class o is important, so we will stress the fact that the singular values depend on o by denoting them as $\sigma_i^o(g)$.

The singular value decomposition provides a map $a : \mathrm{PSL}(d, \mathbb{R}) \to \mathfrak{a}^+$, called the *Cartan projection*, such that for every $g \in \mathrm{PSL}(d, \mathbb{R})$ there exist $k_g, l_g \in K^o$ such that

$$g = k_g \exp(a(g)) l_g.$$

More precisely,

$$a(g) = (\log \sigma_1^o(g), \ldots, \log \sigma_d^o(g)).$$

Recall from Section 2.2 that g has a gap of index p if $\sigma_p^o(g) > \sigma_{p+1}^o(g)$. In that case

$$U_p^o(g) = k_g(u_1 \oplus \cdots \oplus u_p) \qquad (7.2)$$

(note again the dependence on o).

Given $\alpha_p \in \Pi$ denote by $K^o(\{\alpha_p\})$ the stabilizer in K^o of the vector space $u_1 \oplus \cdots \oplus u_p$. Moreover, given a subset $\theta \subset \Pi$, denote

$$K^o(\theta) = \bigcap_{\alpha_p \in \theta} K^o(\{\alpha_p\}).$$

If for some $p \in \{1, \ldots, d\}$ and $g \in \mathrm{PSL}(d, \mathbb{R})$ one has $a_1(g) = a_p(g) > a_{p+1}(g)$, then any element of $k_g K^o(\{\alpha_p\})$ can be chosen in a Cartan decomposition of g. If all the gaps of g are indexed by a subset $\theta \subset \Pi$, then k_g is only defined modulo $K^o(\theta)$ and one has the *partial flag*

$$U^o(g) = \{U_p^o(g) : \alpha_p \in \theta\}.$$

Note that the Cartan projection of g^{-1} is simply $a(g^{-1}) = (-a_d(g), \ldots, -a_1(g))$. The linear transformation $\mathrm{i} : \mathfrak{a} \to \mathfrak{a}$ defined by

$$\mathrm{i}(a_1, \ldots, a_d) = (-a_d, \ldots, -a_1)$$

is called *the opposition involution*. If g has gaps indexed by θ then g^{-1} has gaps indexed by $\mathrm{i}\theta = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}$. Denote $S^o(g) = U^o(g^{-1})$.

7.7. The symmetric space

Recall that fixing a class o defines a splitting

$$\mathfrak{sl}(d, \mathbb{R}) = \mathfrak{p}^o \oplus \mathfrak{k}^o.$$

This splitting is orthogonal with respect to the *Killing form*, the symmetric bilinear form κ on $\mathfrak{sl}(d, \mathbb{R})$ defined by

$$\kappa(A, B) = 2d \operatorname{Trace}(AB).$$

This linear form is related to the adjoint involution $^{\mathsf{t}}$: the linear form $\kappa(\cdot, \cdot^{\mathsf{t}})$ is positive definite.

Since $\mathfrak{p}^o$ consists of fixed points for $^{\mathsf{t}}$, the restriction of κ to $\mathfrak{p}^o$, denoted by $(\cdot, \cdot)_o$: $\mathfrak{p}^o \times \mathfrak{p}^o \to \mathbb{R}$, is positive definite. Explicitly, if $v \in \mathfrak{p}^o$ then v is diagonalizable and

$$|v|_o^2 := (v, v)_o$$

equals the sum of the squared eigenvalues of v.

The space

$$X_d = \{\text{inner products on } \mathbb{R}^d\}/\mathbb{R}_+$$

is a contractible $\operatorname{PSL}(d, \mathbb{R})$-homogeneous space, the action being given by

$$g \cdot \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle' \quad \text{where} \quad \langle v, v \rangle' = \langle g^{-1}v, g^{-1}w \rangle.$$

The stabilizer of o is the group K^o and thus the orbit map $(g, o) \mapsto g \cdot o$ identifies the tangent space $\mathsf{T}_o X_d$ with the vector space $\mathfrak{p}^o$. A direct computation shows that the Riemannian metric $o \mapsto (\cdot, \cdot)_o$ is $\operatorname{PSL}(d, \mathbb{R})$-invariant.

The space $(X_d, (\cdot, \cdot)_o)$ is, by definition, *the symmetric space of* $\operatorname{PSL}(d, \mathbb{R})$.

7.8. Maximal flats

A direct computation shows that the orbit $\exp \mathfrak{a} \cdot o \subset X_d$ is isometric to $(\mathfrak{a}, (\cdot, \cdot)_o)$. Moreover, one can show that $\exp \mathfrak{a} \cdot o$ is a *maximal totally geodesic flat* (flat meaning "isometric to a Euclidean space", maximal with respect to dimension; see for example [Lang, Section XII.3]). For every $g \in \operatorname{PSL}(d, \mathbb{R})$ one has

$$d_{X_d}(o, g \cdot o) = d_{X_d}(o, \exp a(g) \cdot o) = |a(g)|_o = \sqrt{\sum_i (\log \sigma_i^o(g))^2}, \qquad (7.3)$$

where d_{X_d} is the distance on X_d induced by the Riemannian metric $(\cdot, \cdot)_o$.

The translated orbit $g \exp \mathfrak{a} \cdot o$ is again a maximal totally geodesic flat (through $g \cdot o$), and hence all geodesics of X_d are of the form

$$t \mapsto g \exp(ta) \cdot o$$

for a given $g \in \operatorname{PSL}(d, \mathbb{R})$ and $a \in \mathfrak{a}^+$.

In other words, a *maximal flat* in X_d is obtained by fixing a set $\mathcal{L}$ of d lines that span $\mathbb{R}^d$ and considering the space of inner products, up to homothety, that make $\mathcal{L}$ an orthogonal set.

The following lemma is simple but extremely useful for estimations:

Lemma 7.1. *Let $\varphi \in \mathfrak{a}^*$ with $\varphi|\mathfrak{a}^+ - \{0\} > 0$. Then there exists $c > 1$ such that*

$$\frac{1}{c}\varphi(a(g)) \leq d_{X_d}(o, g \cdot o) \leq c\varphi(a(g)) \quad \text{for all } g \in \mathrm{PSL}(d, \mathbb{R}).$$

Proof. Since $\mathfrak{a}^+$ is closed and $\ker \varphi \cap \mathfrak{a}^+ = \{0\}$, the function $a \mapsto |a|_o/\varphi(a)$ is invariant under multiplication by scalars and bounded on $\mathfrak{a}^+ - \{0\}$. Equation (7.3) completes the proof. □

For example, $\log \sigma_1^o(g)$ and $-\log \sigma_d^o(g)$ are comparable to $d_{X_d}(o, g \cdot o)$.

7.9. The Furstenberg boundary and parallel sets

A *parametrized flat* is a function $\mathsf{f}: \mathfrak{a} \to X_d$ of the form

$$\mathsf{f}(a) = g \exp a \cdot o \quad \text{for some } g \in \mathrm{PSL}(d, \mathbb{R}).$$

A maximal flat is thus a subset of the form $\mathsf{f}(\mathfrak{a}) \subset X_d$ for some parametrized flat f.

Observe that $\mathrm{PSL}(d, \mathbb{R})$ acts transitively on the set of parametrized flats and that the stabilizer of $\mathsf{f}_0: a \mapsto \exp a \cdot o$ is the group M. We will hence identify the space of parametrized flats with $\mathrm{PSL}(d, \mathbb{R})/M$.

Two parametrized flats f, g are *equivalent* if the function

$$\mathfrak{a} \ni a \mapsto d_{X_d}(\mathsf{f}(a), \mathsf{g}(a)) \in \mathbb{R}$$

is bounded on $\mathfrak{a}^+$.

The *Furstenberg boundary* of X_d is the space of equivalence classes of parametrized flats. Note that by definition of $N = \exp \mathfrak{n}^+$ the distance function

$$a \mapsto d_{X_d}(n \exp a \cdot o, \exp a \cdot o)$$

is bounded on $\mathfrak{a}^+$ only if $n \in M \exp \mathfrak{a} N = P$.[17] Thus, the equivalence class of the flat f_0 is $P \cdot \mathsf{f}_0$. Hence, the Furstenberg boundary is $\mathrm{PSL}(d, \mathbb{R})$-equivariantly identified with the space $\mathscr{F} = \mathrm{PSL}(d, \mathbb{R})/P$ of complete flags.

Given a parametrized flat f denote by $\mathsf{Z}(\mathsf{f}) \in \mathscr{F}$ its equivalence class in the Furstenberg boundary. Also, denote by $\check{\mathsf{Z}}(\mathsf{f}) \in \mathscr{F}$ the class of the parametrized flat[18]

$$a \mapsto \mathsf{f}(-a).$$

This last identification can be seen directly: a parametrized flat f consists in fixing an *ordered*[19] set $\{\ell_1, \ldots, \ell_d\}$ of d lines that span $\mathbb{R}^d$ and considering all inner products (up to homothety) that make this set an orthogonal set, and the choice of one of these

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[17] The ij entry of $\mathcal{E}$ of $\exp(-ta)n\exp(ta)$ is $\exp(t(a_j - a_i))n_{ij}$. In order to have this entry bounded for all $t > 0$ one must have $n_{ij} = 0$ for all $j < i$.

[18] This is still a parametrized flat.

[19] Recall $\mathfrak{a}^+$ is fixed beforehand.

inner products. The associated point "at infinity" in the Furstenberg boundary of this parametrized flat is the complete flag

$$Z(f)_p = \ell_1 \oplus \cdots \oplus \ell_p.$$

Moreover, $\check{Z}(f)_p = \ell_d \oplus \cdots \oplus \ell_{d-p+1}$.

One easily deduces the following properties:

- Given $x \in X_d$ and a complete flag F there exists a unique maximal flat $f(\mathfrak{a})$ containing x such that $Z(f) = F$: apply the Gram–Schmidt process to flag F and any inner product in the class x.
- Given two flags in general position, $(E, F) \in \mathscr{F}^{(2)}$, there exists a unique maximal flat $f(\mathfrak{a})$ such that $Z(f) = E$ and $\check{Z}(f) = F$: it suffices to consider the ordered set $\ell_p = E_p \cap F_{d-p+1}$.

Recalling that $M \exp \mathfrak{a} = P \cap \check{P}$, observe that the maps $\check{Z}$ and Z are exactly the canonical quotient projections

$$(\check{Z}, Z) \colon \operatorname{PSL}(d, \mathbb{R})/M \to \mathscr{F}^{(2)} = \operatorname{PSL}(d, \mathbb{R})/M \exp \mathfrak{a}.$$

Given subsets $\theta' \subset \theta \subset \Pi$, denote by $\mathscr{F}_\theta$ the space of *partial flags of type θ* and given $E \in \mathscr{F}_\theta$ denote by $E^{\theta'} \in \mathscr{F}_{\theta'}$ the partial flag of type θ' defined by forgetting the irrelevant subspaces of E. Denote by $Z_\theta(f) = Z(f)^\theta$ and by $\check{Z}_\theta = Z_{i\theta}$.

Given a pair of partial flags in general position, $E \in \mathscr{F}_\theta$ and $F \in \mathscr{F}_{i\theta}$, and a point $x \in X_d$, we define:

- The *Weyl cone* $V(x, E)$ determined by x and E is

$$\bigcup_f f(\mathfrak{a}^+),$$

  where the union is over all parametrized flats f with $f(0) = x$ and $Z_\theta(f) = E$.
- The *parallel set* $P(F, E)$ determined by F and E is

$$\bigcup_f f(\mathfrak{a}),$$

  where the union is over all parametrized flats f with $\check{Z}_\theta(f) = F$ and $Z_\theta(f) = E$.

7.10. Parametrized flats through o and $g \cdot o$

Consider $g \in \operatorname{PSL}(d, \mathbb{R})$ and let $g = k_g \exp a(g) l_g$ be a Cartan decomposition of g. Observe that the set of d lines

$$k_g \mathcal{E} = \{k_g u_1, \ldots, k_g u_d\}$$

is simultaneously o-orthogonal and $g \cdot o$-orthogonal. The set of classes of inner products that make this set orthogonal is hence a maximal flat through o and $g \cdot o$.

If g has gaps of certain indices, indexed by $\theta \subset \Pi$, then every element of $k_g K_o(\theta)$ can be chosen in a Cartan decomposition of g. Thus, the set of maximal flats through o and $g \cdot o$ is the translated $K_o(\theta)$-orbit $k_g K_o(\theta)\mathcal{E}$.

All parametrized flats f with $f(0) = o$ and $g \cdot o \in f(\mathfrak{a}^+)$ have $U^o(g)$ as a partial subflag of their corresponding flag $Z(f)$ at infinity (recall (7.2); see Fig. 6).
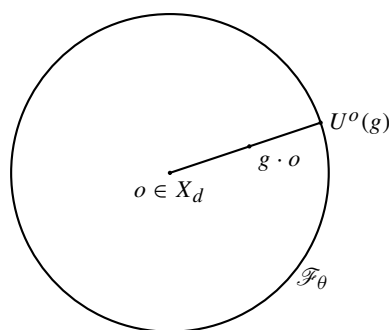
Fig. 6. All flats through o and $g \cdot o$ have $U^o(g)$ as a partial flag at infinity.

7.11. $\mathfrak{a}^+$-valued distance

Recall that X_d is $\mathrm{PSL}(d, \mathbb{R})$-homogeneous and consider the map $\underline{a} \colon X_d \times X_d \to \mathfrak{a}^+$ defined by

$$\underline{a}(g \cdot o, h \cdot o) = a(g^{-1}h).$$

Note that $\underline{a}$ is $\mathrm{PSL}(d, \mathbb{R})$-invariant for the diagonal action of $\mathrm{PSL}(d, \mathbb{R})$ on $X_d \times X_d$, that

$$d_{X_d}(x, y) = |\underline{a}(x, y)|_o \qquad (7.4)$$

(due to (7.3)) and that $\mathrm{i}(\underline{a}(x, y)) = \underline{a}(y, x)$.

Consider the subset of simple roots defined by

$$\theta(x, y) = \{\alpha \in \Pi : \alpha(\underline{a}(x, y)) \neq 0\}$$

and consider the partial flag

$$U(x, y) = \{gU_\alpha^o(g^{-1}h)\}_{\alpha \in \theta(x,y)},$$

where $(x, y) = (g \cdot o, h \cdot o)$. Given $\theta \subset \theta(x, y)$, let the *Weyl cone of type θ from x to y* be

$$V_\theta(x, y) = \bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+),$$

where the union is over all parametrized flats f with $\mathsf{f}(0) = x$ and $\mathsf{Z}_\theta(\mathsf{f}) = U(x, y)^\theta$. Finally, the *diamond of type θ* between x and y is the subset

$$\diamondsuit_\theta(x, y) = V_\theta(x, y) \cap V_{\mathrm{i}\theta}(y, x).$$

This diamond is contained in the parallel set

$$P(U(x, y)^\theta, S(x, y)^{\mathrm{i}\theta}),$$

where $S(x, y) = hS^o(g^{-1}h)$.

For example, consider $x = o$. Any $y \in X_d$ can be written as $y = g \exp a \cdot o$ with $a = \underline{a}(o, y) \in \mathfrak{a}^+$ and $g \cdot o = o$. If $a \in \text{int}\, \mathfrak{a}^+$ then $\theta(o, y) = \Pi$ and

$$V_\Pi(o, y) = \{g \exp v \cdot o : v \in \mathfrak{a}^+\},$$
$$\Diamond_\Pi(o, y) = \{g \exp v \cdot o : v \in \mathfrak{a}^+ \cap (a - \mathfrak{a}^+)\}.$$

7.12. Angles and distances to parallel sets

The purpose of this subsection and the next one is to relate the distance from a given point o to a parallel set $P(E, F)$, for two partial flags in general position, to the angle between E and F for an inner product in the class o (Proposition 7.2).

Fix an inner product $\langle \cdot, \cdot \rangle \in o$ and denote by $\|\cdot\|$ the induced norm on $\mathbb{R}^d$. The *o-angle* between nonzero vectors $v, w \in \mathbb{R}^d$ is defined as the unique number $\angle_o(v, w)$ in $[0, \pi]$ whose cosine is $\langle v, w \rangle / (\|v\| \|w\|)$. If $E, F \subset \mathbb{R}^d$ are nonzero subspaces then we define their *o-angle* as

$$\angle_o(E, F) := \min_{v \in E^\times} \min_{w \in F^\times} \angle_o(v, w), \tag{7.5}$$

where $E^\times := E - \{0\}$. We also write $\angle_o(v, F)$ instead of $\angle_o(\mathbb{R}v, F)$ if v is a nonzero vector. Observe that $\angle_o(\cdot, \cdot)$ is independent of $\langle \cdot, \cdot \rangle \in o$.

We have not found a precise reference for the following proposition, we will hence provide a proof. See Fig. 7.
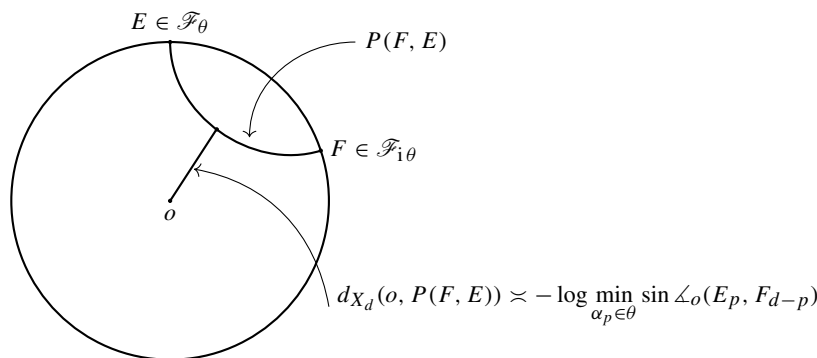

Fig. 7. The statement of Proposition 7.2.

Proposition 7.2. *Given $\theta \subset \Pi$ there exist $c > 1$ and $c' > 0$, only depending on θ and the group $\mathrm{PSL}(d, \mathbb{R})$, such that if $(E, F) \in \mathscr{F}_\theta^{(2)}$, then*

$$\frac{-1}{c} \log \sin \min_{\alpha_p \in \theta} \angle_o(E_p, F_{d-p}) \le d_{X_d}(o, P(F, E)) \le c' - c \log \sin \min_{\alpha_p \in \theta} \angle_o(E_p, F_{d-p}).$$

Proof. Without loss of generality, we may assume that for all $\alpha_p \in \theta$ one has

$$F_{d-p} = u_{d-p} \oplus \cdots \oplus u_d.$$

Denote by $G_{\mathcal{E}}(E, F)$ the set of elements $g \in \mathrm{PSL}(d, \mathbb{R})$ such that for all $\alpha_p \in \theta$ one has $g(u_1 \oplus \cdots \oplus u_p) = E_p$ and $g(F_{d-p}) = F_{d-p}$. Then the parallel set equals

$$P(F, E) = \{g \cdot o : g \in G_{\mathcal{E}}(E, F)\}.$$

Let us prove the first inequality. Consider an arbitrary p with $\alpha_p \in \theta$ and arbitrary nonzero vectors $v \in E_p$ and $w \in F_{d-p}$. Let $\tilde{w}$ be the orthogonal projection of v on the line spanned by w. Let $g \in G_{\mathcal{E}}(E, F)$. Then $g^{-1}v$ and $g^{-1}w$ are orthogonal, which allows us to estimate

$$\frac{1}{\sin \angle_o(v, w)} = \frac{\|\tilde{w}\|}{\|v - \tilde{w}\|} \leq \frac{\sigma_1^o(g)\|g^{-1}\tilde{w}\|}{\sigma_d^o(g)\|g^{-1}v - g^{-1}\tilde{w}\|} \leq \frac{\sigma_1^o(g)}{\sigma_d^o(g)}.$$

By Lemma 7.1, the logarithm of the right-hand side is comparable to $d_{X_d}(o, g \cdot o)$. So we obtain the first inequality in the proposition.

Next, let us turn to the second inequality. Write the set $\{p : \alpha_p \in \theta\} \cup \{0, d\}$ as $\{0 = p_0 < p_1 < \cdots < p_k = d\}$. Denote

$$H_i^0 = u_{p_{i-1}} \oplus \cdots \oplus u_{p_i}, \qquad H_i = E_{p_i} \cap F_{d-p_{i+1}}.$$

For all $i \in \{1, \ldots, k\}$ one has

$$\dim H_1^0 \oplus \cdots \oplus H_i^0 = \dim H_1 \oplus \cdots \oplus H_i.$$

If for $g \in \mathrm{PSL}(d, \mathbb{R})$ one has $g(H_i^0) = H_i$ then $g \in G_{\mathcal{E}}(E, F)$ and hence $g \cdot o \in P(F, E)$. We will define a suitable such g and estimate its operator norm for the class o.

Consider g as the unique element of $\mathrm{PSL}(d, \mathbb{R})$ such that, for each $i = 1, \ldots, k$, the restriction of g^{-1} to H_i coincides with the orthogonal projection on H_i^0.

We proceed to estimate $\|g\|$. Given $v \in \mathbb{R}^d$, write it as $v_1 + \cdots + v_k$ with $v_i \in H_i^0$. Then

$$\|gv\| \leq \sum_{i=1}^k \|gv_i\| = \sum_{i=1}^k \frac{\|v_i\|}{\sin \angle_o(g(v_i), H_1^0 \oplus \cdots \oplus H_{i-1}^0)}.$$

For each i, orthogonality yields $\|v_i\| \leq \|v\|$; moreover,

$$\angle_o(g(v_i), H_1^0 \oplus \cdots \oplus H_{i-1}^0)$$
$$\geq \angle_o(H_i \oplus \cdots \oplus H_d, H_1^0 \oplus \cdots \oplus H_{i-1}^0) = \angle_o(E_{p_i}, F_{d-p_i}) \geq \min_{\alpha_p \in \theta} \angle_o(E_p, F_{d-p}).$$

Therefore

$$\|g\| \leq \frac{d}{\sin \min_{\alpha_p \in \theta} \angle_o(E_p, F_{d-p})}.$$

Taking log and using Lemma 7.1 yields the second inequality. $\qquad\square$

7.13. Regular quasi-geodesics and the Morse Lemma of Kapovich–Leeb–Porti

Let $I \subset \mathbb{Z}$ be an interval and let μ, c be positive numbers. A (μ, c)-*quasi-geodesic* is a map $x \colon I \to X_d$ (also denoted by $\{x_n\}_{n \in I}$) such that

$$\frac{1}{\mu}|n - m| - c \leq d_{X_d}(x_n, x_m) \leq \mu|n - m| + c \quad \text{for all } n, m \in I.$$

Let $\mathscr{C} \subset \mathfrak{a}^+$ be a closed cone. Following [KLP$_2$] we will say that a quasi-geodesic segment $\{x_n\}$ is $\mathscr{C}$-*regular* if $\underline{a}(x_n, x_m) \in \mathscr{C}$ for all $n < m \in I$. Denote

$$\theta_{\mathscr{C}} = \{\alpha \in \Pi : \ker \alpha \cap \mathscr{C} = \{0\}\}.$$

We state the following version of the Morse Lemma [KLP$_2$, Theorem 1.3], specialized to the symmetric space of $\mathrm{PSL}(d, \mathbb{R})$:

Theorem 7.3 (Kapovich–Leeb–Porti [KLP$_2$]). *Let $\mu, c > 0$ and $\mathscr{C} \subset \mathfrak{a}^+$ a closed cone. Then there exists $C > 0$ such that if $\{x_n\}_{n \in I}$ is a $\mathscr{C}$-regular (μ, c)-quasi-geodesic segment, then:*

- *If I is finite then $\{x_n\}$ is at distance at most C from the diamond $\diamondsuit_{\theta_{\mathscr{C}}}(x_{\min I}, x_{\max I})$.*
- *If $I = \mathbb{N}$ then there exists $F \in \mathscr{F}_{\theta_{\mathscr{C}}}$ such that $\{x_n\}$ is contained in the C-neighborhood of the Weyl cone $V(x_{\min I}, F)$.*
- *If $I = \mathbb{Z}$ then there exists $(E, F) \in \mathscr{F}_{\theta_{\mathscr{C}}}^{(2)}$ such that $\{x_n\}$ is contained in the C-neighborhood of $V(z, E) \cup V(z, F)$ for some $z \in P(E, F)$ at uniform distance from $\{x_n\}$.*

Proof. We can assume that $0 \in I$ and $x_0 = o$. Consider $\{h_n\}_{n \in I} \subset \mathrm{PSL}(d, \mathbb{R})$ such that $h_n \cdot o = x_n$. Since $\{x_n\}$ is a quasi-geodesic, equation (7.4) implies

$$|a(h_{n+1}^{-1} h_n)|_o = |\underline{a}(h_n \cdot o, h_{n+1} \cdot o)|_o = d_{X_d}(x_n, x_{n+1}) \leq \mu + c.$$

If we set $g_n = h_{n+1}^{-1} h_n$, then the above implies that $\{g_n\}$ lies in a compact subset of $\mathrm{PGL}(d, \mathbb{R})$. Moreover, if $m \geq n$ then

$$a(g_m \cdots g_n) = a(h_{m+1}^{-1} h_n) = \underline{a}(x_{m+1}, x_n) \in \mathrm{i}\,\mathscr{C}.$$

One has the following:

1. The sequence $\{g_n\}_{n \in I}$ is $\mathrm{i}\,\alpha$-dominated for all $\alpha \in \theta_{\mathscr{C}}$: indeed, since $\mathscr{C}$ is closed and does not intersect $\ker \alpha - \{0\}$, one has $\mathrm{i}\,\mathscr{C} \cap \ker \mathrm{i}\,\alpha = \{0\}$ and hence there exists $\delta > 0$ such that for all $a \in \mathrm{i}\,\mathscr{C} - \{0\}$ and $\alpha \in \theta_{\mathscr{C}}$ one has

$$\mathrm{i}\,\alpha(a) > \delta|a|_o.$$

Thus,

$$\alpha(a(g_m \cdots g_n)) = \alpha(\underline{a}(x_{m+1}, x_n)) > \delta|a(x_{m+1}, x_n)|_o > (\delta/\mu)|n - m| - \delta c.$$

That is, the sequence $\{g_n\}_{n \in I}$ belongs to the space

$$\mathcal{D}(\mu + c, d - p, \delta/\mu, e^{-c\delta}, I)$$

for all p such that $\alpha_p \in \theta_{\mathscr{C}}$, if we use the operator norm $\| \ \|_o$ associated to o.

2. Subsection 7.10 implies that the (partial) flag at infinity associated to the Weyl cone $V_{\theta_{\mathscr{C}}}(o, x_{m+1})$ is

$$
\begin{aligned}
U(o, x_{m+1})^{\theta_{\mathscr{C}}} &= \{U^o_\alpha(h_{m+1})\}_{\alpha \in \theta_{\mathscr{C}}} \\
&= \{U^o_\alpha(g_0^{-1} \cdots g_m^{-1})\}_{\alpha \in \theta_{\mathscr{C}}} = \{S^o_{\mathrm{i}\alpha}(g_m \ldots g_0)\}_{\alpha \in \theta_{\mathscr{C}}} \quad \text{for } m \geq 0, \\
U(o, x_{-m})^{\mathrm{i}\theta_{\mathscr{C}}} &= \{U^o_{\mathrm{i}\alpha}(h_{-m})\}_{\alpha \in \theta_{\mathscr{C}}} = \{U^o_{\mathrm{i}\alpha}(g_0 \cdots g_{-m})\}_{\alpha \in \theta_{\mathscr{C}}} \quad \text{for } m \geq 0.
\end{aligned}
$$

Consider ℓ_1 given simultaneously by Lemmas 2.5 and A.12 for all p such that $\alpha_p \in \theta_{\mathscr{C}}$ and for constants $\mu + c$, δ/μ, and $e^{-c\delta}$. Assume $[-\ell_1, \ell_1] \subset I$ (i.e. I is long enough). Item 2 above and Lemma 2.5 imply the existence of δ_0 such that for all m and $-n$ in I with $m, n \geq \ell_1$ one has

$$
\angle_o(U(o, x_m)^{\theta_{\mathscr{C}}}, U(o, x_{-n})^{\mathrm{i}\theta_{\mathscr{C}}}) > \delta_0.
$$

Moreover, by Lemma A.12 one has $\angle_o(U(o, x_{\ell_1})^{\theta_{\mathscr{C}}}, U(o, x_m)^{\theta_{\mathscr{C}}}) < \varepsilon$ and the same holds for $U(o, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}$ and $U(o, x_{-n})^{\mathrm{i}\theta_{\mathscr{C}}}$.

Proposition 7.2 implies that for all $m, -n \in I$ with $n, m \geq \ell_1$ the distance between o and the parallel set $P(U(o, x_m)^{\theta_{\mathscr{C}}}, U(o, x_{-n})^{\mathrm{i}\theta_{\mathscr{C}}})$ is bounded above by a number C depending on μ, c, the cone $\mathscr{C}$ and a priori the point o, but independent of the quasi-geodesic through o.

Since X_d is $\mathrm{PSL}(d, \mathbb{R})$-homogeneous and this action is by isometries, for any k such that $[k - \ell_1, k + \ell_1] \subset I$ one has

$$
d_{X_d}\big(x_k, P(U(x_k, x_m)^{\theta_{\mathscr{C}}}, U(x_k, x_{-n})^{\mathrm{i}\theta_{\mathscr{C}}})\big) < C \tag{7.6}
$$

provided $m \geq k + \ell_1$ and $-n \leq k - \ell_1$.
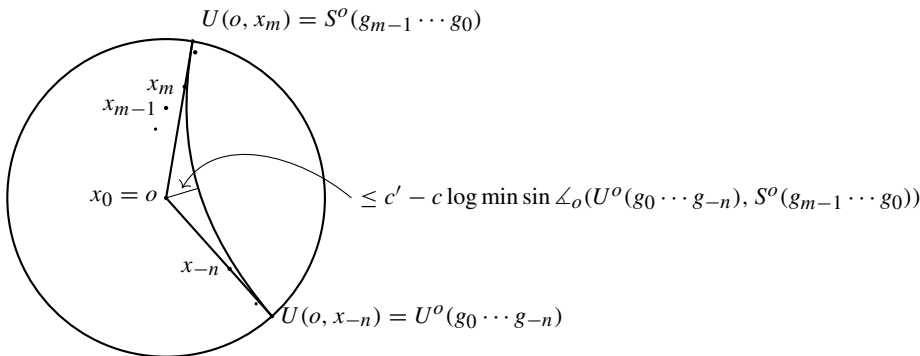

Fig. 8. The flag at infinity associated to the Weyl cone $V_{\theta_{\mathscr{C}}}(o, x_m)$ (resp. $V_{\mathrm{i}\theta_{\mathscr{C}}}(o, x_m)$) corresponds to the S^o flag of a dominated sequence for $m \geq 0$, resp. to the U^o flag when $m \leq 0$.

We now consider the three cases of the statement in turn, in increasing difficulty. **If $I = \mathbb{Z}$,** denote

$$
F = E^{\mathrm{cu}}(\{g_i\}) = \lim_{n \to \infty} U(o, x_{-n})^{\mathrm{i}\theta_{\mathscr{C}}}, \qquad E = E^{\mathrm{cs}}(\{g_i\}) = \lim_{m \to \infty} U(o, x_m)^{\theta_{\mathscr{C}}}.
$$

Recall that $E^{\mathrm{cu}} \oplus E^{\mathrm{cs}}$ is a dominated splitting for the cocycle ϑ (Subsection 2.3), in particular it is equivariant. Hence, if $k > 0$ then

$$h_k^{-1} F = g_k \cdots g_0 E^{\mathrm{cu}}(\{g_i\}_{i \in \mathbb{Z}}) = E^{\mathrm{cu}}(\{g_{i-k}\}_{i \in \mathbb{Z}})$$

and $h_k^{-1} E = g_k \cdots g_0 E^{\mathrm{cs}}(\{g_i\}_{i \in \mathbb{Z}}) = E^{\mathrm{cs}}(\{g_{i-k}\}_{i \in \mathbb{Z}})$. Consequently,

$$\angle_o(h_k^{-1} F, h_k^{-1} E) > \delta.$$

Thus

$$d(h_k \cdot o, P(E, F)) = d(o, P(h_k^{-1} F, h_k^{-1} E)) < C.$$

The same argument works for $k < 0$ and so $\{x_i\}_{i \in \mathbb{Z}}$ is at Hausdorff distance at most C from the parallel set $P(F, E)$.

   If $I = \mathbb{N}$, assume for simplicity that $I = [-\ell_1, \infty) \cap \mathbb{Z}$. The flag $U(o, x_m)^{\theta_{\mathscr{C}}}$ converges as $m \to \infty$ to a flag $E \in \mathscr{F}_{\theta_{\mathscr{C}}}$. We have to show that for all $k > 0$ the x_k angle

$$\angle_{x_k}(U(x_k, o)^{\mathrm{i}\theta_{\mathscr{C}}}, E)$$

is bounded below by a constant independent of k.

   As before, we know that if $k \geq \ell_1$ and $m \geq k + \ell_1$ then

$$\angle_{x_k}\big(U(x_k, o)^{\mathrm{i}\theta_{\mathscr{C}}}, U(x_k, x_m)^{\theta_{\mathscr{C}}}\big) > \delta.$$

For a homothety class $x \in X_d$ denote by d_x the distance on $\mathscr{F}_{\theta_{\mathscr{C}}}$ induced by an inner product in x. We will show that $d_{x_k}(U(x_k, x_m)^{\theta_{\mathscr{C}}}, U(o, x_m)^{\theta_{\mathscr{C}}})$ goes to zero as $m \to \infty$, and thus the angle $\angle_{x_k}(U(x_k, o)^{\mathrm{i}\theta_{\mathscr{C}}}, E)$ will be uniformly bounded below.

   Recall that by definition $U(x_k, x_m)^{\theta_{\mathscr{C}}} = h_k U^o(h_k^{-1} h_m)^{\theta_{\mathscr{C}}}$, hence

$$d_{x_k}\big(U(x_k, x_m)^{\theta_{\mathscr{C}}}, U(o, x_m)^{\theta_{\mathscr{C}}}\big) = d_{h_k \cdot o}\big(h_k U^o(h_k^{-1} h_m)^{\theta_{\mathscr{C}}}, U^o(h_m)^{\theta_{\mathscr{C}}}\big)$$
$$= d_o\big(U^o(h_k^{-1} h_m)^{\theta_{\mathscr{C}}}, h_k^{-1} U^o(h_m)^{\theta_{\mathscr{C}}}\big).$$

Since k is fixed, Lemma A.5 implies $d_o(U^o(h_k^{-1} h_m)^{\theta_{\mathscr{C}}}, h_k^{-1} U^o(h_m)^{\theta_{\mathscr{C}}}) \to 0$ as $m \to \infty$.

   Let us now show that given $\varepsilon > 0$ there exists $L \in \mathbb{N}$ (depending on the quasi-geodesic constants and $\mathscr{C}$) such that if $k \geq L$ then

$$d_{x_k}\big(U(x_k, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, U(o, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}\big) < \varepsilon;$$

this will conclude the proof in this case since every $\{x_i\}_{i \in \mathbb{N}}$ will be at bounded distance from a Weyl cone pointing to E in the parallel set $P(U(o, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, E)$.

   One has

$$d_{x_k}\big(U(x_k, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, U(o, x_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}\big) = d_{h_k \cdot o}\big(h_k U^o(h_k^{-1} h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, U^o(h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}\big)$$
$$= d_o\big(U^o(h_k^{-1} h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, h_k^{-1} U^o(h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}\big).$$

Since

$$\angle_o\big(U^o(h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, S^o(h_k^{-1})^{\mathrm{i}\theta_{\mathscr{C}}}\big) = \angle_o\big(U^o(h_{-\ell_1})^{\mathrm{i}\theta_{\mathscr{C}}}, U^o(h_k)^{\theta_{\mathscr{C}}}\big) > \delta,$$

Lemmas A.6 and A.9 imply the desired conclusion.

If I is finite, notice that in the last paragraph we have proved the following: given $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that for all $k \in I$ with $k - L \geq \min I$ one has

$$d_{x_k}\big(U(x_k, x_{\min I})^{\mathrm{i}\theta_{\mathscr{C}}}, U(x_{\min I+L}, x_{\min I})^{\mathrm{i}\theta_{\mathscr{C}}}\big) < \varepsilon.$$

Analogously, for every $k \in I$ with $k + L \leq \max I$ one has

$$d_{x_k}\big(U(x_k, x_{\max I})^{\theta_{\mathscr{C}}}, U(x_{\max I-L}, x_{\max I})^{\theta_{\mathscr{C}}}\big) < \varepsilon.$$

Hence, for every k with $\min I + L < k < \max I - L$ one has

$$\angle_{x_k}\big(U(x_{\min I+L}, x_{\min I})^{\mathrm{i}\theta_{\mathscr{C}}}, U(x_{\max I-L}, x_{\max I})^{\theta_{\mathscr{C}}}\big)$$
$$\geq \angle_{x_k}\big(U(x_k, x_{\min I})^{\mathrm{i}\theta_{\mathscr{C}}}, U(x_k, x_{\max I})^{\theta_{\mathscr{C}}}\big) - 2\varepsilon > \delta - 2\varepsilon.$$

Choosing ε such that $\delta - 2\varepsilon > 0$ completes the proof. $\qquad\square$

8. When the target group is a semisimple Lie group

The purpose of this section is to extend the main results of the previous sections to the situation where the target group is a real-algebraic semisimple Lie group without compact factors.

We will begin by recalling the general structure theory of such groups, needed to define concepts such as domination, Anosov representation, regular quasi-geodesic, etc. This basic structure theory can be found in [Hum$_1$], [Ebe], [Hel].

We will then explain how the representation theory of these groups is used to reduce the general case to the PSL$(d, \mathbb{R})$ case, for a well chosen d. Section 7 mimics, for PSL$(d, \mathbb{R})$, the general structure presented here.

The first main goal is Subsection 8.5, where Theorem 3.2 and Proposition 4.9 are extended to the general setting. This general case is reduced to the actual statement of 3.2 and 4.9 using Tits representations.

The remainder of the section is devoted to a new proof of the Morse Lemma of [KLP$_2$] for symmetric spaces of noncompact type. In contrast with Subsection 8.5, a simple reduction to the PSL$(d, \mathbb{R})$ case is not sufficient: one needs to have a finer control of distances to parallel sets when embedding symmetric spaces. This is achieved in Corollary 8.8.

If G is a Lie group with Lie algebra $\mathfrak{g}$, the *Killing form* of $\mathfrak{g}$ is the symmetric bilinear form defined by

$$\kappa(v, w) = \mathrm{Trace}(\mathrm{ad}_v\, \mathrm{ad}_w).$$

The group G is *semisimple* if κ is nondegenerate.

We will assume from now on that G is a semisimple, real-algebraic (i.e. defined by polynomial equations with real coefficients) Lie group, without compact factors (i.e. there is no normal subgroup H of G such that G/H is compact).

8.1. Root system

A *Cartan involution* of $\mathfrak{g}$ is a Lie algebra morphism $o \colon \mathfrak{g} \to \mathfrak{g}$ with $o^2 = 1$ and such that the bilinear form $(v, w) \mapsto -\kappa(v, o(w))$ is positive definite. The fixed point set

$$\mathfrak{k}^o = \{v \in \mathfrak{g} : ov = v\}$$

is the Lie algebra of a maximal compact subgroup K^o. Let $\mathfrak{p}^o = \{v \in \mathfrak{g} : ov = -v\}$ and note that

$$\mathfrak{g} = \mathfrak{k}^o \oplus \mathfrak{p}^o.$$

A computation shows that $[\mathfrak{p}^o, \mathfrak{p}^o] \subset \mathfrak{k}^o$ and hence any subalgebra of $\mathfrak{p}^o$ is necessarily abelian. Let $\mathfrak{a} \subset \mathfrak{p}^o$ be a maximal abelian subalgebra.

Denote by Σ the set of *restricted roots* of $\mathfrak{a}$ on $\mathfrak{g}$. By definition,

$$\Sigma = \left\{\alpha \in \mathfrak{a}^* - \{0\} : \mathfrak{g}_\alpha \neq 0\right\}$$

where

$$\mathfrak{g}_\alpha = \{w \in \mathfrak{g} : [a, w] = \alpha(a)w \,\, \forall a \in \mathfrak{a}\}.$$

The closure of a connected component of

$$\mathfrak{a} - \bigcup_{\alpha \in \Sigma} \ker \alpha$$

is called a *closed Weyl chamber*. Fix a closed Weyl chamber $\mathfrak{a}^+$ and let $\Sigma^+ = \{\alpha \in \Sigma : \alpha|_{\mathfrak{a}^+} \geq 0\}$ be the set of *positive roots* associated to $\mathfrak{a}^+$. The set Σ^+ contains a subset Π such that

- Π is a basis of $\mathfrak{a}$ as a vector space,
- every element of Σ^+ has nonnegative coefficients in the basis Π.

The set Π is called the *set of simple (restricted) roots* determined by Σ^+, and the sets $\ker \alpha \cap \mathfrak{a}^+$ for $\alpha \in \Pi$ are the *walls* of the chamber $\mathfrak{a}^+$.

The *Weyl group* W of Σ is defined as the group generated by the orthogonal reflections in the subspaces $\{\ker \alpha : \alpha \in \Sigma\}$.

The reflections associated to elements of Π span W. With respect to the word-length on this generating set, there exists a unique longest element in W, denoted by $u_0 \colon \mathfrak{a} \to \mathfrak{a}$. This is the unique element in W that sends $\mathfrak{a}^+$ to $-\mathfrak{a}^+$. The *opposition involution* $\mathrm{i} \colon \mathfrak{a} \to \mathfrak{a}$ is defined by $\mathrm{i} = -u_0$. Denote by $(\cdot, \cdot)$ the bilinear form on $\mathfrak{a}^*$ dual to the Killing form, define

$$\langle \chi, \psi \rangle = \frac{2(\chi, \psi)}{(\psi, \psi)} \quad \text{for } \chi, \psi \in \mathfrak{a}^*,$$

and let $\{\omega_\alpha\}_{\alpha \in \Pi}$ be the *dual basis* of Π, i.e. $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha\beta}$. The linear form ω_α is the *fundamental weight* associated to α. Note that

$$\chi = \sum_{\alpha \in \Pi} \langle \chi, \alpha \rangle \, \omega_\alpha \quad \text{for every } \chi \in \mathfrak{a}^*. \tag{8.1}$$

Denote by $a = a_G \colon G \to \mathfrak{a}^+$ the *Cartan projection* of G. By definition, for every $g \in G$ one has $g \in K^o \exp a(g) K^o$ and $a(g^{-1}) = \mathrm{i}\, a(g)$.

8.2. Parabolic subgroups

Denote by M the centralizer of $\exp \mathfrak{a}$ in K and let $N = \exp \mathfrak{n}^+$ where $\mathfrak{n}^+ = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$. The group $P_\Pi = M \exp \mathfrak{a}\, N$ is called a *minimal parabolic subgroup* and its Lie algebra is $\mathfrak{p}_\Pi = \mathfrak{g}_0 \oplus \mathfrak{n}^+$. A *parabolic subgroup* of G is a subgroup that contains a conjugate of P_Π. Two parabolic subgroups are *opposite* if their intersection is a reductive group.[20]

To each subset θ of Π one associates two opposite parabolic subgroups of G, P_θ and $\check{P}_\theta$, whose Lie algebras are, by definition,

$$\mathfrak{p}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_{-\alpha}, \quad \check{\mathfrak{p}}_\theta = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha} \oplus \bigoplus_{\alpha \in \langle \Pi - \theta \rangle} \mathfrak{g}_\alpha,$$

where $\langle \theta \rangle$ is the set of positive roots generated by θ. Every pair of opposite parabolic subgroups of G is conjugate to $(P_\theta, \check{P}_\theta)$ for a unique θ, and every opposite parabolic subgroup of P_θ is conjugate to $P_{\mathrm{i}\theta}$, the parabolic group associated to

$$\mathrm{i}\theta = \{\alpha \circ \mathrm{i} : \alpha \in \theta\}.$$

The quotient space $\mathscr{F} = \mathscr{F}_\Pi = G/P_\Pi$ is called the *flag space* of G and if $\theta \subset \Pi$ then $\mathscr{F}_\theta = G/P_\theta$ is called a *partial flag space* of G. Note that if $\theta \subset \theta' \subset \Pi$ one has $P_{\theta'} \subset P_\theta$ and there is hence a canonical projection $\mathscr{F}_{\theta'} \to \mathscr{F}_\theta$, denoted by $x \mapsto x^\theta$.

Finally, denote by $\mathscr{F}_\theta^{(2)} \subset \mathscr{F}_\theta \times \mathscr{F}_{\mathrm{i}\theta}$ the space of pairs of opposite parabolic subgroups (of type θ); this is the unique open G-orbit on $\mathscr{F}_\theta \times \mathscr{F}_{\mathrm{i}\theta}$.

8.3. Representations of G

Let $\Lambda \colon G \to \mathrm{PSL}(V)$ be a finite-dimensional rational[21] irreducible representation and denote by $\phi_\Lambda \colon \mathfrak{g} \to \mathfrak{sl}(V)$ the Lie algebra homomorphism associated to Λ. Then $\chi \in \mathfrak{a}^*$ is a *restricted weight* of Λ if the vector space

$$V_\chi = \{v \in V : \phi_\Lambda(a)v = \chi(a)v \ \forall a \in \mathfrak{a}\}$$

is nonzero. Theorem 7.2 of Tits [Ti] states that the set of weights has a unique maximal element with respect to the order $\chi \geq \psi$ if $\chi - \psi$ is positive on $\mathfrak{a}^+$. This is called *the highest weight* of Λ and denoted by χ_Λ.

Note that if χ is a restricted weight and $v \in V_\chi$ then, for $n \in \mathfrak{g}_\alpha$ with $\alpha \in \Sigma$, $\phi_\Lambda(n)v$ is an eigenvector of $\phi_\Lambda(\mathfrak{a})$ of eigenvalue[22] $\chi + \alpha$ unless $\phi_\Lambda(n)v = 0$. Since for all $\beta \in \Sigma^+$ one has $\chi_\Lambda + \beta \geq \chi_\Lambda$ and χ_Λ is maximal, one concludes that $\chi_\Lambda + \beta$ is not a weight, i.e. for all $n \in \mathfrak{g}_\beta$ and $v \in V_{\chi_\Lambda}$ one has $\phi_\Lambda(n)v = 0$.

Let $\theta_\Lambda \subset \Pi$ be the set of simple roots α such that $\chi_\Lambda - \alpha$ is still a weight of Λ.

---

[20] Recall that a Lie group is *reductive* if its Lie algebra splits into a semisimple algebra and an abelian algebra.

[21] That is, a rational map between algebraic varieties.

[22] Indeed, this follows from $\phi_\Lambda([a, n])v = \alpha(a)\phi_\Lambda(n)v$.

Remark 8.1. The subset θ_Λ is the smallest subset of simple roots such that the following holds. Consider $\alpha \in \Sigma^+$, $n \in \mathfrak{g}_{-\alpha}$ and $v \in V_{\chi_\Lambda}$. Then $\phi_\Lambda(n)v = 0$ if and only if $\alpha \in \langle \Pi - \theta_\Lambda \rangle$. Equivalently, the smallest parabolic subgroup P of G stabilizing V_{χ_Λ} is of type θ_Λ.

Remark 8.2. Since V is irreducible, the $\phi_\Lambda(\mathfrak{a})$-eigenspaces

$$\Big(\prod_{\alpha \in \Pi} \phi_\Lambda(n_\alpha^{i_\alpha})\Big) V_{\chi_\Lambda}$$

span V and thus any other weight of Λ is of the form

$$\chi_\Lambda - \sum_{\alpha \in \Pi} k_\alpha \alpha, \tag{8.2}$$

where $k_\alpha \geq 0$ and $\sum_{\alpha \in \theta_\Lambda} k_\alpha \neq 0$ (i.e. the numbers k_α for $\alpha \in \theta_\Lambda$ do not simultaneously vanish).

Consider an inner product on V invariant under ΛK^o such that $\Lambda \exp \mathfrak{a}$ is symmetric, and denote by Λo its homothety class. For the Euclidean norm $\| \ \|$ induced by this scalar product, one has

$$\log \|\Lambda g\| = \chi_\Lambda(a(g)). \tag{8.3}$$

If $g = k(\exp a(g))l$ with $k, l \in K^o$, then for all $v \in l^{-1} V_{\chi_\Lambda}$ one has

$$\|\Lambda g(v)\| = \|\Lambda g\| \, \|v\|.$$

If we denote $\ell = \dim V_{\chi_\Lambda}$ then Remark 8.2 implies that

$$\alpha_\ell(a(\Lambda g)) = a_\ell(\Lambda g) - a_{\ell+1}(\Lambda g) = \beta(a(g)) \quad \text{for all } g \in G \tag{8.4}$$

for some $\beta \in \theta_\Lambda$ (depending on g).

Denote by W_{χ_Λ} the $\Lambda(\exp \mathfrak{a})$-invariant complement of V_{χ_Λ}. The space W_{χ_Λ} is Λo-orthogonal to V_{χ_Λ}. Its stabilizer (in G) is opposite to P, and hence conjugate to $P_{i\theta_\Lambda}$. Thus, one has a map of flag spaces

$$(\xi_\Lambda, \xi_\Lambda^*) \colon \mathscr{F}_{\theta_\Lambda}^{(2)} \to \mathscr{F}_{\{\alpha_p\}}^{(2)}. \tag{8.5}$$

This is a proper embedding which is a homeomorphism onto its image.

8.4. A set of representations defined by Tits

One has the following proposition by Tits (see also Humphreys [Hum$_2$, Chapter XI]).

Proposition 8.3 (Tits [Ti]). *For each $\alpha \in \Pi$ there exists a finite-dimensional rational irreducible representation $\Lambda_\alpha \colon G \to \mathrm{PSL}(V_\alpha)$ such that χ_{Λ_α} is an integer multiple of the fundamental weight ω_α and $\dim V_{\chi_{\Lambda_\alpha}} = 1$.*

Such a set of representations is not necessarily unique but will be fixed from now on, and we will say that Λ_α is the *Tits representation* of G associated to α. Observe that (8.4) implies that

$$a_1(\Lambda_\alpha g) - a_2(\Lambda_\alpha g) = \alpha(a(g)) \quad \text{for all } g \in G. \tag{8.6}$$

8.5. Dominated representations: reduction to the $\mathrm{GL}(d, \mathbb{R})$ case

A representation $\rho : \Gamma \to G$ is θ-*dominated* if there exist positive constants μ and c such that

$$\alpha(a(\rho\gamma)) \geq \mu|\gamma| - c \quad \text{for all } \alpha \in \theta \text{ and } \gamma \in \Gamma.$$

Assume that ρ is a θ-dominated representation. Then for all $\alpha \in \theta$, the representation $\Lambda_\alpha \rho : \Gamma \to \mathrm{PSL}(V_\alpha)$ is 1-dominated in the sense of Subsection 3.1. Indeed, (8.6) implies that

$$\log \sigma_1(\Lambda_\alpha \rho(\gamma)) - \log \sigma_2(\Lambda_\alpha \rho(\gamma)) = \alpha(a(\rho\gamma)) \geq \mu|\gamma| - c,$$

or equivalently

$$\frac{\sigma_2(\Lambda_\alpha \rho(\gamma))}{\sigma_1(\Lambda_\alpha \rho(\gamma))} \leq e^c e^{-\mu|\gamma|}.$$

Thus, Theorem 3.2 implies that Γ is word-hyperbolic. Moreover, Proposition 4.9 together with [GW, Proposition 4.3] implies that ρ is P_θ-Anosov.[23]

Thus, Theorem 3.2, Proposition 4.9 together with [GW, Proposition 4.3] and Tits representations 8.3 prove the following.

Theorem 8.4. *Let $\rho : \Gamma \to G$ be a θ-dominated representation. Then Γ is word-hyperbolic and ρ is P_θ-Anosov.*

8.6. A Plücker representation

Given $\theta \subset \Pi$ one can construct a rational irreducible representation of G such that P_θ is the stabilizer of a line, and hence $\check{P}_\theta$ will be the stabilizer of a hyperplane.[24] More precisely, one has the following result from representation theory (see [GW, §4]).

Proposition 8.5. *Given $\theta \subset \Pi$, there exists a finite-dimensional rational irreducible representation $\Lambda : G \to \mathrm{PSL}(V)$ and $\ell \in \mathbb{P}(V)$ such that*

$$P_\theta = \{g \in G : \Lambda g(\ell) = \ell\}.$$

Such a representation can be defined as follows: if we denote $k = \dim \mathfrak{p}_\theta$ then the composition $\Lambda^k \mathrm{Ad} : G \to \mathrm{PSL}(\Lambda^k \mathfrak{g})$ satisfies the desired conditions, except (maybe) irreducibility; this is fixed by considering the vector space V spanned by the G-orbit of the line $\ell = \Lambda^k \mathfrak{p}_\theta$,

$$V = \langle \Lambda^k \mathrm{Ad}\, G \cdot \ell \rangle,$$

and considering the restriction of $\Lambda^k \mathrm{Ad}\, G$ to V.

A representation as in the statement of Proposition 8.5 will be called a *Plücker representation* of G associated to θ. For such a representation one has a continuous equivariant map $(\xi, \xi^*) : \mathscr{F}_\theta^{(2)} \to \mathbb{P}^{(2)}(V)$ which is a homeomorphism onto its image.

---

[23] See [Lab] or [GW] for a precise definition.

[24] There are actually infinitely many such representations.

8.7. The symmetric space

The *symmetric space of G* is the space of Cartan involutions on $\mathfrak{g}$, and is denoted by X. This a G-homogeneous space, and the stabilizer of $o \in X$ is the compact group K^o, whose Lie algebra is $\mathfrak{k}^o$. The tangent space $T_o X$ is hence identified with $\mathfrak{p}^o$. The G-invariant Riemannian metric on X is the restriction of the Killing form κ to $\mathfrak{p}^o \times \mathfrak{p}^o$.

If d_X is the distance on X induced by $\kappa|\mathfrak{p}^o \times \mathfrak{p}^o$, then the Euclidean norm $\| \ \|_o$ induced on $\mathfrak{a}$ is invariant under the Weyl group, and for all $a \in \mathfrak{a}$ one has $d_X(o, (\exp a) \cdot o) = \|a\|_o$. Hence the Cartan decomposition of G implies that

$$d_X(o, g \cdot o) = \|a(g)\|_o \quad \text{for all } g \in G.$$

Consider the map $\underline{a} : X \times X \to \mathfrak{a}^+$ defined by $\underline{a}(g \cdot o, h \cdot o) = a(g^{-1}h)$. Note that $\underline{a}$ is G-invariant for the diagonal action of G on $X \times X$, that

$$d_X(p, q) = \|\underline{a}(p, q)\|_o \tag{8.7}$$

and that $\mathrm{i}(\underline{a}(p, q)) = \underline{a}(q, p)$.

8.8. Flats

A *parametrized flat* is a map $\mathsf{f} : \mathfrak{a} \to X$ of the form $\mathsf{f}(v) = g \exp(v) \cdot o$ for some $g \in G$. Observe that G acts transitively on the set of parametrized flats and the stabilizer of $\mathsf{f}_0 : v \mapsto \exp(v) \cdot o$ is the group M of elements in K commuting with $\exp(\mathfrak{a})$. We will hence identify the space of parametrized flats with G/M.

Two parametrized flats f, g are *equivalent* if the function

$$\mathfrak{a} \ni v \mapsto d_X(\mathsf{f}(v), \mathsf{g}(v)) \in \mathbb{R}$$

is bounded on $\mathfrak{a}^+$. The *Furstenberg boundary* of X is the space of equivalence classes of parametrized flats. It is a standard fact that this space is G-equivariantly identified with $\mathscr{F} = G/P_\Pi$, thus the Furstenberg boundary will also be denoted by $\mathscr{F}$. Denote by

$$\mathsf{Z} : \{\text{parametrized flats}\} \to \mathscr{F}$$

the canonical projection, and set $\check{\mathsf{Z}}(\mathsf{f}) = \mathsf{Z}(\mathsf{f} \circ \mathrm{i})$. The pair $(\check{\mathsf{Z}}(\mathsf{f}), \mathsf{Z}(\mathsf{f}))$ belongs to $\mathscr{F}^{(2)}$.

The following proposition is standard.

Proposition 8.6 (see [GJT, Chapter III]). (i) *A pair $(p, x) \in X \times \mathscr{F}$ determines a unique parametrized flat f such that $\mathsf{f}(0) = p$ and $\mathsf{Z}(\mathsf{f}) = x$.*
(ii) *A point $(x, y) \in \mathscr{F}^{(2)}$ determines a unique maximal flat $\mathsf{f}_{xy}(\mathfrak{a})$ such that $\check{\mathsf{Z}}(\mathsf{f}_{xy}) = x$ and $\mathsf{Z}(\mathsf{f}_{xy}) = y$.*

Given a subset $\theta \subset \Pi$ of simple roots and a pair $(x, y) \in \mathscr{F}_\theta^{(2)}$ of partial flags in general position, the *parallel set $P(x, y)$ from x to y* is

$$\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}),$$

where the union is over all parametrized flats f with $\check{\mathsf{Z}}(\mathsf{f})^\theta = x$ and $\mathsf{Z}(\mathsf{f})^{\mathrm{i}\theta} = y$.

8.9. Representations and distances to parallel sets

The purpose of this subsection is to deduce the following proposition and corollary. Only the statement of Corollary 8.8 will be needed in what follows. If $\Lambda : G \to \mathrm{PSL}(V)$ is a finite-dimensional rational irreducible representation, denote by $\Lambda : X \to X_V$ the induced map between symmetric spaces and by $D\Lambda : \mathsf{T}X \to \mathsf{T}X_V$ its differential mapping. The map $D_p\Lambda$ is $\phi_\Lambda|\mathfrak{p}^p : \mathfrak{p}^p \to \mathfrak{p}^{\Lambda p}$.

Proposition 8.7. *Let $\Lambda : G \to \mathrm{PSL}(V)$ be a finite-dimensional rational irreducible representation. Then there exists a constant $\delta > 0$ such that if $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ and $p \in P(x, y)$ then*

$$\measuredangle\big(D_p\Lambda(\mathsf{T}_p P(x, y)^\perp), \mathsf{T}_{\Lambda p} P(\xi_\Lambda x, \xi_\Lambda^* y)\big) > \delta,$$

where $\measuredangle$ denotes the angle on $\mathsf{T}_{\Lambda p} X_V$.

Corollary 8.8. *Let $\Lambda : G \to \mathrm{PSL}(V)$ be an injective finite-dimensional irreducible representation. Then there exists $c > 0$ such that if $o \in X$ and $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$, then*

$$\frac{1}{c} d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)) \le d_X(o, P(x, y)) \le c\, d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)).$$

Let us show how Proposition 8.7 implies the corollary.

Proof of Corollary 8.8. Since G has no compact factors, X is non-positively-curved. Hence, since $P(x, y)$ is totally geodesic, the distance from o to $P(x, y)$ is attained at a unique point $p \in P(x, y)$. Moreover, the geodesic segment $\sigma : I \to X$ from p to o is orthogonal to $P(x, y)$ at p, i.e. $\dot\sigma(0) \in \mathsf{T}_p P(x, y)^\perp$. Similarly, the distance from Λo to $P(\xi_\Lambda x, \xi_\Lambda^* y)$ is also attained at a unique point q and the geodesic segment from q to Λo is perpendicular to $P(\xi_\Lambda x, \xi_\Lambda^* y)$.

Since ΛX is totally geodesic in X_V, we can estimate the angle at the vertex Λp of the geodesic triangle $\{\Lambda o, \Lambda p, q\}$. Indeed, Proposition 8.7 implies that this angle is bounded below by $\delta > 0$, independently of o and (x, y). Since the angle at the vertex q is $\pi/2$, a CAT(0) argument completes the proof. $\qquad\square$

The remainder of this subsection is devoted to the proof of Proposition 8.7.

Proof of Proposition 8.7. Let $L(x, y)$ be the stabilizer in G of $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ and let $\mathfrak{l}$ be its Lie algebra. Moreover, let $L(\xi_\Lambda x, \xi_\Lambda^* y)$ be the stabilizer in $\mathrm{PSL}(V)$ of $(\xi_\Lambda x, \xi_\Lambda^* y)$, and $\mathfrak{l}' \subset \mathfrak{sl}(V)$ its Lie algebra.

If $\mathfrak{l}^\perp$ denotes the orthogonal of $\mathfrak{l}$ with respect to the Killing form κ, then we will show that

$$\phi_\Lambda(\mathfrak{l}^\perp) \cap \mathfrak{l}' = \{0\}. \tag{8.8}$$

This will imply the proposition since if $p \in P(x, y)$ then $\mathsf{T}_p P(x, y) = \mathfrak{p}^p \cap \mathfrak{l}$, so its orthogonal in $\mathsf{T}_p X$ is $\mathfrak{p}^p \cap \mathfrak{l}^\perp$ and given $q \in P(x, y)$ there exists $g \in L(x, y)$ such that $gp = q$; thus the angle

$$\measuredangle(\mathfrak{p}^{\Lambda p} \cap \phi_\Lambda(\mathfrak{l}^\perp), \mathfrak{p}^{\Lambda p} \cap \mathfrak{l}') \ge \measuredangle(\phi_\Lambda(\mathfrak{p}^p \cap \mathfrak{l}^\perp), \mathfrak{p}^{\Lambda p} \cap \mathfrak{l}')$$

is independent of p in $P(x, y)$.

Now we turn to showing that (8.8) holds. By homogeneity, we can assume that the stabilizer of x is P_{θ_Λ} and the stabilizer of y is $\check{P}_{\theta_\Lambda}$. The parallel set $P(x, y)$ is then the orbit $L_{\theta_\Lambda} \cdot o$, where L_{θ_Λ} is the Levi group $P_\theta \cap \check{P}_\theta$.

An explicit computation shows that for $\alpha, \beta \in \Sigma$, the eigenspaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ (recall Subsection 7.1) are orthogonal with respect to the Killing form κ whenever $\alpha \neq -\beta$. If we denote $\mathfrak{g}_{\{\alpha,-\alpha\}} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ then the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

is orthogonal with respect to κ.

Note that the Cartan involution o sends $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$ and hence $\mathfrak{g}_{\{\alpha,-\alpha\}}$ is o-invariant. Since $-\kappa(\cdot, o(\cdot))$ is positive definite and $\kappa|\mathfrak{g}_\alpha = 0$ one concludes that $\mathfrak{p}^o \cap \mathfrak{g}_{\{\alpha,-\alpha\}} \neq \{0\}$. One then finds an orthogonal decomposition

$$\mathsf{T}_o X = \mathfrak{p}^o = \mathfrak{a} \oplus \bigoplus_{\alpha \in \Sigma^+} \mathfrak{p}^o \cap \mathfrak{g}_{\{\alpha,-\alpha\}}. \tag{8.9}$$

The Lie algebra of L_{θ_Λ} is

$$\mathfrak{l}_{\theta_\Lambda} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

and hence the decomposition

$$\mathfrak{g} = \mathfrak{l}_{\theta_\Lambda} \oplus \bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}$$

is orthogonal with respect to κ, that is,

$$\mathfrak{l}_{\theta_\Lambda}^\perp = \bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{\{\alpha,-\alpha\}}. \tag{8.10}$$

Set $\ell = \dim V_{\chi_\Lambda}$ and note that $\mathfrak{l}'$ is the Lie algebra of the stabilizer of $(V_{\chi_\Lambda}, W_{\chi_\Lambda}) \in \mathscr{F}_{\{\alpha_\ell\}}^{(2)}$. Choose a Cartan algebra $\mathfrak{a}_\mathcal{E}$ of $\mathfrak{sl}(V)$ and a Weyl chamber $\mathfrak{a}_\mathcal{E}^+$ such that[25] (recall Section 8.2):

- $\mathfrak{p}_{\{\alpha_\ell\}}$ is the Lie algebra of the parabolic group stabilizing V_{χ_Λ},
- $\check{\mathfrak{p}}_{\{\alpha_\ell\}}$ is the Lie algebra of the parabolic group stabilizing W_{χ_Λ}.

The Lie algebra $\mathfrak{l}'$ is thus $\mathfrak{p}_{\{\alpha_\ell\}} \cap \check{\mathfrak{p}}_{\{\alpha_\ell\}}$. In order to show that $\phi_\Lambda(\mathfrak{l}_{\theta_\Lambda}^\perp) \cap \mathfrak{l}' = \{0\}$ the following lemma is sufficient.

Lemma 8.9. *The subspace* $\phi_\Lambda\left(\bigoplus_{\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle} \mathfrak{g}_{-\alpha}\right)$ *has trivial intersection with* $\mathfrak{p}_{\{\alpha_\ell\}}$.

---

[25] It suffices to choose a Λo-orthogonal set of $\dim V$ lines $\mathcal{E}$ such that $\mathcal{E} \subset V_{\chi_\Lambda} \cup W_{\chi_\Lambda}$.

Proof. Recall that

$$\mathfrak{p}_{\{\alpha_p\}} = \mathfrak{a}_\mathcal{E} \oplus \bigoplus_{\beta \in \Sigma_V^+} \mathfrak{sl}(V)_\beta \oplus \bigoplus_{\beta \in \langle \Pi_V - \{\alpha_l\} \rangle} \mathfrak{sl}(V)_{-\beta},$$

where Σ_V^+ and Π_V are the sets of positive, resp. simple, roots associated to the choice of $\mathfrak{a}_\mathcal{E}^+$.

Consider then $\alpha \in \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle$ and $n \in \mathfrak{g}_{-\alpha}$. Remark 8.1 implies that if $v \in V_{\chi_\Lambda}$ then $\phi_\Lambda(n)v \neq 0$. Hence,

$$\phi_\Lambda(n) \notin \mathfrak{a}_\mathcal{E} \oplus \bigoplus_{\beta \in \Sigma_V^+} \mathfrak{sl}(V)_\beta \oplus \bigoplus_{\beta \in \langle \Pi_V - \{\alpha_l\} \rangle} \mathfrak{sl}(V)_{-\beta}.$$

Let $\{\gamma_i\}_{i=1}^k \subset \Sigma^+ - \langle \Pi - \theta_\Lambda \rangle$ be pairwise distinct and consider $n_{\gamma_i} \in \mathfrak{g}_{-\gamma_i} - \{0\}$. Then $\phi_\Lambda(n_{\gamma_i})v$ is a nonzero eigenvector of $\mathfrak{a}$ of eigenvalue $\chi - \gamma_i$, hence

$$\phi_\Lambda(n_{\gamma_1} + \cdots + n_{\gamma_k})v = \phi_\Lambda(n_{\gamma_1})v + \cdots + \phi_\Lambda(n_{\gamma_k})v \neq 0.$$

This proves the lemma. $\qquad\square$

As an example, consider the irreducible representation (unique up to conjugation) $\Lambda :$ $\mathrm{PSL}(2, \mathbb{R}) \to \mathrm{PSL}(3, \mathbb{R})$. Explicitly Λ is the action of $\mathrm{PSL}(2, \mathbb{R})$ on the symmetric power $\mathbf{S}^2(\mathbb{R}^2)$, which is a 3-dimensional space spanned by

$$\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_2\},$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

The group $L(\langle e_1 \rangle, \langle e_2 \rangle)$ has Lie algebra $\mathfrak{l} = \mathfrak{a} = \left\langle \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix} \right) \right\rangle$, its orthogonal with respect to the Killing form of $\mathrm{PSL}(2, \mathbb{R})$ is

$$\mathfrak{a}^\perp = \left\langle \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix} \right) \right\rangle.$$

The group $L(\langle e_1 \otimes e_1 \rangle, \langle e_1 \otimes e_2, e_2 \otimes e_2 \rangle)$ has Lie algebra

$$\mathfrak{l}' = \left\langle \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{smallmatrix} \right) \right\rangle.$$

The orthogonal complement of $\mathfrak{l}'$ with respect to the Killing form of $\mathrm{PSL}(3, \mathbb{R})$ is

$$\mathfrak{l}'^\perp = \left\langle \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right) \right\rangle.$$

An explicit computation shows that $\Lambda \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 1 & t & t^2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{smallmatrix} \right)$, hence

$$\phi_\Lambda \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix} \right) = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix} \right),$$

which does not belong to $\mathfrak{l}'^\perp$. However, it does not belong to $\mathfrak{l}'$ either, which gives the definite angle of Proposition 8.7. $\qquad\square$

8.10. Gromov product and representations

Consider $\theta \subset \Pi$ and denote by

$$\mathfrak{a}_\theta = \bigcap_{\alpha \in \Pi - \theta} \ker \alpha$$

the Lie algebra of the center of the reductive group $L_\theta = P_\theta \cap \check{P}_\theta$. Recall from [Sa] that the *Gromov product*[26] based on $o \in X$ is the map

$$(\cdot | \cdot)_o : \mathscr{F}_\theta^{(2)} \to \mathfrak{a}_\theta$$

defined by the unique vector $(x|y)_o \in \mathfrak{a}_\theta$ such that

$$\omega_\alpha((x|y)_o) = -\log \sin \angle_{\Lambda_\alpha o}(\xi_{\Lambda_\alpha} x, \xi^*_{\Lambda_\alpha} y) = -\log \frac{|\varphi(v)|}{\|\varphi\|_{\Lambda_\alpha o} \|v\|_{\Lambda_\alpha o}} \quad \text{for all } \alpha \in \theta,$$

where ω_α is the fundamental weight of Λ_α, $v \in \xi_{\Lambda_\alpha} x - \{0\}$ and $\ker \varphi = \xi^*_{\Lambda_\alpha} y$.

Remark 8.10. Note that

$$\max_{\alpha \in \theta} \omega_\alpha((x|y)_o) = -\log \min_{\alpha \in \theta} \sin \angle_{\Lambda_\alpha o}(\xi_{\Lambda_\alpha} x, \xi^*_{\Lambda_\alpha} y). \tag{8.11}$$

Note also that since $\{\omega_\alpha|\mathfrak{a}_\theta\}_{\alpha \in \theta}$ is a basis of $\mathfrak{a}_\theta$, the right-hand side of (8.11) is comparable to the norm $\|(x|y)_o\|_o$.

As suggested by the notation, the Gromov product is independent of the choice of Tits's representations of G, moreover, it keeps track of Gromov products for all irreducible representations of G. Indeed, one has the following consequence of (8.3).

Remark 8.11. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a finite-dimensional rational irreducible representation with $\dim V_{\chi_\Lambda} = 1$. If $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ then[27]

$$(\xi_\Lambda x | \xi^*_\Lambda y)_{\Lambda o} = \chi_\Lambda((x|y)_o) = \sum_{\alpha \in \theta_\Lambda} \langle \chi_\Lambda, \alpha \rangle \, \omega_\alpha((x|y)_o).$$

Note that, by definition of θ_Λ (recall Subsection 8.3), the coefficients in the last equation are all strictly positive.

8.11. Gromov product and distances to parallel sets

The aim of this subsection is to prove the following.

Proposition 8.12. *Given $\theta \subset \Pi$ there exist $c > 1$ and $c' > 0$ only depending on G such that*

$$\frac{1}{c} \|(x|y)_o\|_o \leq d_X(o, P(x, y)) \leq c \|(x|y)_o\|_o + c' \quad \text{for all } (x, y) \in \mathscr{F}_\theta^{(2)}.$$

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[26] This is the negative of the one defined in [Sa].

[27] We have identified $\mathfrak{a}_{\{\alpha_1\}}$ with $\mathbb{R}$ via ω_{α_1}.

Proof. As for $\mathrm{PSL}(d, \mathbb{R})$, the first inequality follows easily. Let us show the second one. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a Plücker representation of G associated to the set θ (recall Subsection 8.6). Corollary 8.8 implies that there exists c_0 such that for all $o \in X$ and $(x, y) \in \mathscr{F}_{\theta_\Lambda}^{(2)}$ one has

$$d_X(o, P(x, y)) \leq c_0 d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)).$$

Note that $\xi_\Lambda x \in \mathbb{P}(V)$ and that $\xi_\Lambda^* y \in \mathbb{P}(V^*)$ Proposition 7.2 applied to the set $\{\alpha_1\} \subset \Pi_V$ implies that

$$d_{X_V}(\Lambda o, P(\xi_\Lambda x, \xi_\Lambda^* y)) \leq c(\xi_\Lambda x | \xi_\Lambda^* y)_{\Lambda o} + c'.$$

Finally, Remark 8.11 states that

$$(\xi_\Lambda x | \xi_\Lambda^* y)_{\Lambda o} = \sum_{\alpha \in \theta} \langle \chi_\Lambda, \alpha \rangle \, \omega_\alpha((x|y)_o),$$

where $\langle \chi_\Lambda, \alpha \rangle > 0$ for all $\alpha \in \theta$, so the last quantity is comparable to $\|(x|y)_o\|$. $\qquad \square$

8.12. *The Morse Lemma of Kapovich–Leeb–Porti*

Let us begin with some definitions. Consider $p, q \in X$, $\theta \subset \Pi$ and $(x, y) \in \mathscr{F}_\theta^{(2)}$.

- The *Weyl cone* $V(p, x)$ through p and x is $\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+)$, where the union is over all parametrized flats f with $\mathsf{f}(0) = p$ and $Z(\mathsf{f})^\theta = x$.
- Let
$$\theta(p, q) = \{\alpha \in \Pi : \alpha(\underline{a}(p, q)) \neq 0\}$$
and let $U(p, q) \in \mathscr{F}_{\theta(p,q)}$ be the flag determined by $Z(\mathsf{f})^{\theta(p,q)} = U(p, q)$ for every parametrized flat with $\mathsf{f}(0) = p$ and $q \in \mathsf{f}(\mathfrak{a}^+)$.
- If $\theta \subset \theta_{(p,q)}$ the *Weyl cone* $V_\theta(p, q)$ through p and q (in that order) is

$$\bigcup_{\mathsf{f}} \mathsf{f}(\mathfrak{a}^+),$$

where the union is over all parametrized flats f with $\mathsf{f}(0) = p$, and $Z(\mathsf{f})^\theta = U(p, q)^\theta$.
- Finally, if $\theta \subset \theta_{(p,q)}$ the θ-*diamond* between p and q is the subset

$$\diamondsuit_\theta(p, q) = V_\theta(p, q) \cap V_{\mathsf{i}\theta}(q, p).$$

If $\mathscr{C} \subset \mathfrak{a}^+$ is a closed cone, consider the subset

$$\theta_\mathscr{C} = \{\alpha \in \Pi : \ker \alpha \cap \mathscr{C} = \{0\}\}.$$

Let $I \subset \mathbb{Z}$ be an interval. Following [KLP$_2$] we will say that a quasi-geodesic segment $\{p_n\}_{n \in I} \subset X$ is $\mathscr{C}$-*regular* if

$$\underline{a}(p_n, p_m) \in \mathscr{C} \quad \text{for all } n < m.$$

One has the following version of the Morse Lemma.

Theorem 8.13 ([KLP$_2$, Theorem 1.3]). *Let $\mu, c > 0$ and $\mathscr{C} \subset \mathfrak{a}^+$ a closed cone. Then there exists $C > 0$ such that if $\{p_n\}_{n \in I}$ is a $\mathscr{C}$-regular (μ, c)-quasi-geodesic segment, then:*

- *If I is finite then $\{p_n\}$ is at distance at most C from $\diamondsuit_{\theta_\mathscr{C}}(p_{\min I}, p_{\max I})$.*
- *If $I = \mathbb{N}$ then there exists $x \in \mathscr{F}_{\theta_\mathscr{C}}$ such that $\{p_n\}$ is contained in the C-neighborhood of the Weyl cone $V_{\theta_\mathscr{C}}(p_{\min I}, x)$.*
- *If $I = \mathbb{Z}$ then there exist two partial flags $(x, y) \in \mathscr{F}_{\theta_\mathscr{C}}^{(2)}$ in general position such that $\{p_n\}$ is contained in the C-neighborhood of the union $V_{\theta_\mathscr{C}}(z, x) \cup V_{\mathrm{i}\,\theta_\mathscr{C}}(z, y)$ for some $z \in P(x, y)$ at uniform distance from $\{p_n\}$.*

Proof. Let $\Lambda : G \to \mathrm{PSL}(V)$ be a Plücker representation associated to $\theta_\mathscr{C}$. If $\{p_n\}$ is a $\mathscr{C}$-regular quasi-geodesic then $\{\Lambda p_n\}$ is a $\Lambda\mathscr{C}$-regular quasi-geodesic. Moreover, (8.4) implies that the cone $\Lambda\mathscr{C}$ does not intersect the wall $\ker \alpha_1$.

The proof now follows the lines of the proof of Theorem 7.3, with the use of Proposition 8.12. □

Appendix. Auxiliary technical results

In this appendix we collect a number of lemmas that are used elsewhere in the paper. These lemmas are either quantitative linear-algebraic facts or properties of dominated splittings. Certainly many of these results are known, but they do not necessarily appear in the literature in the exact form or setting that we need; therefore we include proofs for the reader's convenience.

A.1. Angles, Grassmannians

The *angle* between nonzero vectors $v, w \in \mathbb{R}^d$ is defined as the unique number $\angle(v, w)$ in $[0, \pi]$ whose cosine is $\langle v, w \rangle / (\|v\| \|w\|)$. If $P, Q \subset \mathbb{R}^d$ are nonzero subspaces then we define their *angle* as

$$\angle(P, Q) := \min_{v \in P^\times} \min_{w \in Q^\times} \angle(v, w), \tag{A.1}$$

where $P^\times := P - \{0\}$. We also write $\angle(v, Q)$ instead of $\angle(\mathbb{R}v, Q)$ if v is a nonzero vector.

Given integers $1 \le p < d$, we let $\mathscr{G}_p(\mathbb{R}^d)$ be the *Grassmannian* formed by the p-dimensional subspaces of $\mathbb{R}^d$. Let us metrize $\mathscr{G}_p(\mathbb{R}^d)$ in a convenient way. If $p = 1$, the sine of the angle defines a distance. (We leave it to the reader to check the triangle inequality.) In general, we may regard each element of $\mathscr{G}_p(\mathbb{R}^d)$ as a compact subset of $\mathscr{G}_1(\mathbb{R}^d)$, and then use the Hausdorff metric induced by the distance on $\mathscr{G}_1(\mathbb{R}^d)$. In other words, we define, for $P, Q \in \mathscr{G}_p(\mathbb{R}^d)$,

$$d(P, Q) := \max\left\{ \max_{v \in P^\times} \min_{w \in Q^\times} \sin \angle(v, w),\ \max_{w \in Q^\times} \min_{v \in P^\times} \sin \angle(v, w) \right\}. \tag{A.2}$$

Note the trivial bound

$$d(P, Q) \geq \sin \angle(P, Q). \tag{A.3}$$

In fact, the following stronger fact holds: for any subspace $\tilde{Q} \subset \mathbb{R}^d$ (not necessarily of dimension p),

$$\tilde{Q} \cap Q \neq \{0\} \implies d(P, Q) \geq \sin \angle(P, \tilde{Q}). \tag{A.4}$$

Actually the two quantities between curly brackets in (A.2) coincide, that is,

$$d(P, Q) = \max_{w \in Q^\times} \min_{v \in P^\times} \sin \angle(v, w) = \max_{w \in Q^\times} \sin \angle(w, P). \tag{A.5}$$

This follows from the existence of an isometry of $\mathbb{R}^d$ that interchanges P and Q (see [Wo, Theorem 2]).

As the reader can easily check, relation (A.5) can be rewritten in the following ways:

$$d(P, Q) = \max_{w \in Q^\times} \max_{u \in (P^\perp)^\times} \cos \angle(u, w) \tag{A.6}$$

$$= \max_{w \in Q^\times} \min_{v \in P} \frac{\|v - w\|}{\|w\|}, \tag{A.7}$$

where $P^\perp$ denotes the orthogonal complement of P. As a consequence of (A.1) and (A.6),

$$d(P, Q) = \cos \angle(P^\perp, Q). \tag{A.8}$$

Another expression for the distance $d(P, Q)$ is given in (A.26) below.

A finer description of the relative position of a pair of elements in the Grassmannian is given by the next proposition:

Proposition A.1 (Canonical angles between a pair of spaces). *Let $P, Q \in \mathscr{G}_p(\mathbb{R}^d)$. Then there exist*

- *numbers $\beta_1 \geq \cdots \geq \beta_p \in [0, \pi/2]$, called* canonical angles, *and*
- *orthonormal bases $\{v_1, \ldots, v_p\}$ and $\{w_1, \ldots, w_p\}$ of P and Q respectively*

such that, for all $i, j \in \{1, \ldots, p\}$,

$$\angle(v_i, w_j) = \begin{cases} \beta_i & \text{if } i = j, \\ \pi/2 & \text{if } i \neq j, \end{cases}$$

and moreover

- *if $p > d/2$ then $\beta_{d-p+1} = \beta_{d-p+2} = \cdots = \beta_p = 0$;*
- *$d(P, Q) = \sin \beta_1$.*

The proof is a simple application of the singular value decomposition [Ste, p. 73].

A.2. More about singular values

If $A\colon E \to F$ is a linear map between inner product vector spaces then we define its *norm* and its *conorm* (or *mininorm*) by

$$\|A\| := \max_{v \in E^\times} \frac{\|Av\|}{\|v\|}, \quad \mathbf{m}(A) := \min_{v \in E^\times} \frac{\|Av\|}{\|v\|}.$$

The following properties hold whenever they make sense:

$$\|AB\| \le \|A\| \|B\|, \quad \mathbf{m}(AB) \ge \mathbf{m}(A)\mathbf{m}(B), \quad \mathbf{m}(A) = \|A^{-1}\|^{-1}.$$

In terms of singular values, we have $\|A\| = \sigma_1(A)$ and $\mathbf{m}(A) = \sigma_d(A)$, where $d = \dim E$.

For convenience, let us assume that A is a linear map from $\mathbb{R}^d$ to $\mathbb{R}^d$. We have the following useful "minimax" characterization of the singular values:

$$\sigma_p(A) = \max_{P \in \mathscr{G}_p(\mathbb{R}^d)} \mathbf{m}(A|_P), \tag{A.9}$$

$$\sigma_{p+1}(A) = \min_{Q \in \mathscr{G}_{d-p}(\mathbb{R}^d)} \|A|_Q\|; \tag{A.10}$$

see [Ste, Cor. 4.30]. Moreover, if A has a gap of index p (that is, $\sigma_p(A) > \sigma_{p+1}(A)$) then the maximum and the minimum above are respectively attained (uniquely) at the spaces $P = S_{d-p}(A)^\perp$ and $Q = S_{d-p}(A)$ (defined in §2.2).

A.3. Linear-algebraic lemmas

In this subsection we collect a number of estimates that will be useful later. Fix integers $1 \le p \le d$.

Lemma A.2. *Let* $A, B \in \mathrm{GL}(d, \mathbb{R})$. *Then*

$$\max\{\mathbf{m}(A)\sigma_p(B), \sigma_p(A)\mathbf{m}(B)\} \le \sigma_p(AB) \le \min\{\|A\|\sigma_p(B), \sigma_p(A)\|B\|\}.$$

Proof. The two inequalities follow from (A.9) and (A.10) respectively. □

Lemma A.3. *Let* $A \in \mathrm{GL}(d, \mathbb{R})$ *have a gap of index* p. *Then, for all unit vectors* $v, w \in \mathbb{R}^d$,

$$\|Av\| \ge \sigma_p(A) \sin \angle(v, S_{d-p}(A)), \tag{A.11}$$

$$\|A^{-1}w\| \ge \sigma_{p+1}(A)^{-1} \sin \angle(w, U_p(A)). \tag{A.12}$$

Also, for all $Q \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ *and* $P \in \mathscr{G}_p(\mathbb{R}^d)$,

$$\|A|_Q\| \ge \sigma_p(A)d(Q, S_{d-p}(A)), \tag{A.13}$$

$$\|A^{-1}|_P\| \ge \sigma_{p+1}(A)^{-1}d(P, U_p(A)). \tag{A.14}$$

Proof. Given a unit vector $v \in \mathbb{R}^d$, decompose it as $v = s + u$ where $s \in S_{d-p}(A)$ and $u \in S_{d-p}(A)^\perp$. Then $\|u\| = \sin \angle(v, S_{d-p}(A))$. Moroever, since As and Au are orthogonal, we have $\|Av\| \geq \|Au\| \geq \sigma_p(A)\|u\|$. This proves (A.11), from which (A.13) follows. Inequalities (A.12) and (A.14) follow from the previous ones applied to the matrix A^{-1}, which has a gap of index $d - p$. $\quad\square$

The next three lemmas should be thought of as follows: if A has a strong gap of index p and $\|B^{\pm 1}\|$ are not too large, then $U_p(AB)$ is close to $U_p(A)$, and $U_p(BA)$ is close to $B(U_p(A))$; moreover, $A(P)$ is close to $U_p(A)$ for any $P \in \mathscr{G}_p(\mathbb{R}^d)$ whose angle with $S_{d-p}(A)$ is not too small.[28]

Lemma A.4. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$. If A and AB have gaps of index p then*

$$d(U_p(A), U_p(AB)) \leq \|B\| \|B^{-1}\| \frac{\sigma_{p+1}(A)}{\sigma_p(A)}.$$

For another way to estimate the distance of $U_p(A)$ and $U_p(AB)$ that does not rely on "smallness" of B, see Lemma A.9 below.

Proof of Lemma A.4. We have

$$
\begin{aligned}
d(U_p(AB), U_p(A)) &\leq \sigma_{p+1}(A)\|A^{-1}|_{U_p(AB)}\| &&\text{(by (A.14))} \\
&\leq \sigma_{p+1}(A)\|B\| \|B^{-1}A^{-1}|_{U_p(AB)}\| \\
&= \sigma_{p+1}(A)\|B\|\sigma_p(AB)^{-1} \\
&\leq \sigma_{p+1}(A)\|B\| \|B^{-1}\|\sigma_p(A)^{-1} &&\text{(by Lemma A.2).} \quad\square
\end{aligned}
$$

Lemma A.5. *Let $A, B \in \mathrm{GL}(d, \mathbb{R})$. If A and BA have gaps of index p then*

$$d\big(B(U_p(A)), U_p(BA)\big) \leq \|B\| \|B^{-1}\| \frac{\sigma_{p+1}(A)}{\sigma_p(A)}.$$

Proof. We have

$$
\begin{aligned}
d\big(B(U_p(A)), U_p(BA)\big) &\leq \sigma_{p+1}(BA)\|A^{-1}B^{-1}|_{B(U_p(A))}\| &&\text{(by (A.14))} \\
&\leq \|B\|\sigma_{p+1}(A)\|A^{-1}|_{U_p(A)}\| \|B^{-1}\| &&\text{(by Lemma A.2)} \\
&= \|B\|\sigma_{p+1}(A)\sigma_p(A)^{-1}\|B^{-1}\|. &&\quad\square
\end{aligned}
$$

Lemma A.6. *Let $A \in \mathrm{GL}(d, \mathbb{R})$ have a gap of index p. Then, for all $P \in \mathscr{G}_p(\mathbb{R}^d)$ transverse to $S_{d-p}(A)$,*

$$d(A(P), U_p(A)) \leq \frac{\sigma_{p+1}(A)}{\sigma_p(A)} \frac{1}{\sin \angle(P, S_{d-p}(A))}.$$

---

[28] We remark that angle estimates of this flavor appear in the usual proofs of the Oseledets theorem: see e.g. [Sim, pp. 141–142]. We also note that [GGKW, Lemma 5.8] contains a generalization of Lemmas A.4 and A.6 to more general Lie groups.

Proof. By (A.14),

$$d(A(P), U_p(A)) \leq \sigma_{p+1}(A)\|A^{-1}|_{A(P)}\| = \frac{\sigma_{p+1}(A)}{\mathbf{m}(A|_P)}.$$

By (A.11), $\mathbf{m}(A|_P) \geq \sigma_p(A) \sin \angle(P, S_{d-p}(A))$, so the lemma is proved. $\qquad\square$

The next lemma implies that the singular values of a product of matrices are approximately the products of the singular values, provided that certain angles are not too small:

Lemma A.7. *Let* $A, B \in \mathrm{GL}(d, \mathbb{R})$. *Suppose that* A *and* AB *have gaps of index* p. *Let* $\alpha := \angle(U_p(B), S_{d-p}(A))$. *Then*

$$\sigma_p(AB) \geq (\sin \alpha)\sigma_p(A)\sigma_p(B),$$
$$\sigma_{p+1}(AB) \leq (\sin \alpha)^{-1}\sigma_{p+1}(A)\sigma_{p+1}(B).$$

Proof. By (A.9) we have

$$\sigma_p(AB) \geq \mathbf{m}(AB|_{B^{-1}(U_p(B))}) \geq \mathbf{m}(A|_{U_p(B)})\mathbf{m}(B|_{B^{-1}(U_p(B))}) = \mathbf{m}(A|_{U_p(B)})\sigma_p(B).$$

On the other hand, inequality (A.11) yields $\mathbf{m}(A|_{U_p(B)}) \geq (\sin \alpha)\sigma_p(A)$, and so we obtain the first inequality in the lemma. The second inequality follows from the first one and the fact that $\sigma_{p+1}(A^{-1}) = 1/\sigma_{d-p}(A)$. $\qquad\square$

A.4. A sketch of the proof of Theorem 2.2

Note that due to the uniqueness property of dominated splittings (Proposition 2.1), it is sufficient to prove the theorem for $\mathbb{T} = \mathbb{Z}$, which is done in [BG]. For the convenience of the reader, let us include here a summary of that proof, using some lemmas that we have already proved in §A.3.

As the "only if" part is not difficult, let us consider the "if" part. So assume that the gap between the p-th and the $(p + 1)$-th singular values of ψ_x^n increases uniformly exponentially with time n. Fix any $x \in X$. Using Lemma A.4, we see that the distance between consecutive elements of the sequence $U_p(\psi_{\phi^{-n}(x)}^n)$ in $\mathcal{G}_p(E_x)$ decreases exponentially fast, and in particular the sequence has a limit E_x^{cu}. Uniform control on the speed of convergence shows that E^{cu} is a continuous subbundle of E. Lemma A.5 implies that this subbundle is invariant.[29] Analogously we obtain the subbundle E^{cs}.

To conclude the proof, we need to show that the bundles E^{cu} and E^{cs} are transverse, and that the resulting splitting is indeed dominated. Here, [BG] uses an ergodic-theoretical argument: The gap between singular values implies that for any Lyapunov regular point x, the difference $\lambda_p(x) - \lambda_{p+1}(x)$ between the p-th and the $(p + 1)$-th Lyapunov exponents is bigger than some constant $2\varepsilon > 0$. Moreover, the Oseledets theorem implies that E_x^{cu} and E_x^{cs} are sums of the Oseledets spaces corresponding to Lyapunov exponents $\geq \lambda_p(x)$ and $\leq \lambda_{p+1}(x)$ respectively. Bearing these facts in mind, assume for a contradiction that

---

[29] This step is missing in [BG].

E^{cu} does not dominate E^{cs}. Then there exist points $x_i \in X$, unit vectors $v_i \in E^{\mathrm{cs}}_{x_i}$ and $w_i \in E^{\mathrm{cu}}_{x_i}$, and times $n_i \to \infty$ such that

$$\frac{\|\psi^{n_i}(v_i)\|}{\|\psi^{n_i}(w_i)\|} > e^{-\varepsilon n_i}.$$

It follows from a Krylov–Bogolyubov argument[30] (making use of the continuity of the subbundles; see [BG] for details) that there exists a Lyapunov regular point x such that $\lambda_p(x) - \lambda_{p+1}(x) \leq \varepsilon$. This contradiction establishes domination.

A.5. Exterior powers and applications

We recall quantitative facts about exterior powers; all the necessary information can be found in [Ar, §3.2.3].

The space $\Lambda^p \mathbb{R}^d$ is endowed with the inner product defined on decomposable p-vectors as

$$\langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle := \det(\langle v_i, w_j \rangle)_{i,j=1,\dots,p}. \tag{A.15}$$

Geometrically (see [Ga, §IX.5]), $\|v_1 \wedge \cdots \wedge v_p\|$ is the p-volume of the parallelepiped with edges $v_1, \dots, v_p$. Therefore, if those vectors span a p-dimensional space $P \subset \mathbb{R}^d$ then, for any $A \in \mathrm{GL}(d, \mathbb{R})$,

$$\frac{\|(\Lambda^p A)(v_1 \wedge \cdots \wedge v_p)\|}{\|v_1 \wedge \cdots \wedge v_p\|} = \mathrm{jac}(A|_P), \tag{A.16}$$

where $\mathrm{jac}(\cdot)$ denotes the *jacobian* of a linear map, i.e., the product of its singular values.

The *Plücker embedding* is the map $\iota \colon \mathscr{G}_p(\mathbb{R}^d) \to \mathscr{G}_1(\Lambda^p \mathbb{R}^d)$ such that if P is spanned by $v_1, \dots, v_p \in \mathbb{R}^d$ then $\iota(P)$ is spanned by $v_1 \wedge \cdots \wedge v_p \in \Lambda^p \mathbb{R}^d$. We metrize $\mathscr{G}_1(\Lambda^p \mathbb{R}^d)$ as described in Subsection A.1.

Lemma A.8. *For all $P, Q \in \mathscr{G}_p(\mathbb{R}^d)$ we have*

$$d(\iota(P), \iota(Q)) \geq d(P, Q), \tag{A.17}$$

$$\sin \angle(\iota(P), \iota(Q)^\perp) \geq [\sin \angle(P, Q^\perp)]^{\min\{p, d-p\}}. \tag{A.18}$$

Proof. Let $\beta, \hat{\beta} \in [0, \pi/2]$ be such that

$$\sin \beta = d(P, Q) \quad \text{and} \quad \sin \hat{\beta} = d(\iota(P), \iota(Q)) = \angle(\iota(P), \iota(Q)).$$

Recalling (A.8), we have

$$\cos \beta = \angle(P, Q^\perp) \quad \text{and} \quad \cos \hat{\beta} = \angle(\iota(P), \iota(Q)^\perp).$$

---

[30] That is, convergence (in the weak star topology) of measures supported in long segments of orbits to an invariant measure.

Associated to P, Q, Proposition A.1 provides canonical angles $\beta_1 \geq \cdots \geq \beta_p$, where $\beta_1 = \beta$, and canonical orthonormal bases $\{v_1, \ldots, v_p\}$ and $\{w_1, \ldots, w_p\}$ for P and Q respectively. Using (A.15) we compute

$$\cos\hat{\beta} = \cos\angle(\iota(P), \iota(Q)) = \langle v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p \rangle = \prod_{i=1}^{p} \cos\beta_i.$$

Then, on the one hand, $\cos\hat{\beta} \leq \cos\beta$, that is, $\sin\hat{\beta} \geq \sin\beta$, which is estimate (A.17). On the other hand, since at most $n := \min\{p, d-p\}$ canonical angles are nonzero, we have $\cos\hat{\beta} \geq (\cos\beta)^n$, which is estimate (A.18). $\qquad\square$

Given $A \in \mathrm{GL}(d, \mathbb{R})$, the singular values of $\Lambda^p A$ are the numbers $\sigma_{i_1}(A) \cdots \sigma_{i_p}(A)$, where $1 \leq i_1 < \cdots < i_p \leq d$; in particular,

$$\sigma_1(\Lambda^p A) = \sigma_1(A) \cdots \sigma_{p-1}(A)\sigma_p(A), \tag{A.19}$$

$$\sigma_2(\Lambda^p A) = \sigma_1(A) \cdots \sigma_{p-1}(A)\sigma_{p+1}(A). \tag{A.20}$$

So A has a gap of index p if and only if $\Lambda^p A$ has a gap of index 1. In this case, the corresponding singular spaces are related via the Plücker embedding $\iota : \mathscr{G}_p(\mathbb{R}^d) \to \mathscr{G}_1(\Lambda^p \mathbb{R}^d)$ as follows:

$$U_1(\Lambda^p A) = \iota(U_p(A)), \tag{A.21}$$

$$S_{\binom{d}{p}-1}(\Lambda^p A) = \iota(S_{d-p}(A)^\perp)^\perp. \tag{A.22}$$

The following result is used at the end of Section 7; it should be compared with Lemma A.4.

Lemma A.9. *Suppose that* $A, B, AB \in \mathrm{GL}(d, \mathbb{R})$ *have gaps of index p, and that the spaces* $S_{d-p}(A)$ *and* $U_p(B)$ *are transverse. Then*

$$d(U_p(AB), U_p(A)) \leq \frac{\sigma_{p+1}(A)}{\sigma_p(A)}[\sin\angle(U_p(B), S_{d-p}(A))]^{-\min\{p, d-p\}}.$$

Proof. First we consider the case $p = 1$. Let v and w be unit vectors spanning the spaces $B^{-1}A^{-1}U_1(AB)$ and $B^{-1}U_1(B)$ respectively. By inequality (A.11),

$$\frac{\|ABw\|}{\|Bw\|} \geq \sigma_1(A) \sin\angle(Bw, S_{d-1}(A)),$$

while by inequality (A.12),

$$\frac{\|Bv\|}{\|ABv\|} = \frac{\|A^{-1}ABv\|}{\|ABv\|} \geq \sigma_2(A)^{-1} \sin\angle(ABv, U_1(B)).$$

Multiplying the two inequalities, we obtain

$$\frac{\sigma_1(A)}{\sigma_2(A)}d(U_1(AB), U_1(B)) \sin\angle(U_1(B), S_{d-1}(A)) \leq \frac{\|ABw\|}{\|ABv\|} \times \frac{\|Bv\|}{\|Bw\|} \leq 1 \times 1 = 1,$$

which is the desired inequality in the case $p = 1$.

Now consider arbitrary p. If A, B, and AB have gaps of index p then the corresponding exterior powers have gaps of index 1, and by the previous case we have

$$d\big(U_1(\Lambda^p(AB)), U_p(\Lambda^p A)\big) \leq \frac{\sigma_2(\Lambda^p A)}{\sigma_1(\Lambda^p A)}\big[\sin \angle\big(U_1(\Lambda^p B), S_{\binom{d}{p}-1}(\Lambda^p A)\big)\big]^{-1}$$

By (A.19) and (A.20), we have

$$\frac{\sigma_2(\Lambda^p A)}{\sigma_1(\Lambda^p A)} = \frac{\sigma_{p+1}(A)}{\sigma_p(A)}.$$

By (A.21) and (A.17),

$$d\big(U_1(\Lambda^p(AB)), U_p(\Lambda^p A)\big) = d\big(\iota(U_p(AB)), \iota(U_p(A))\big) \geq d(U_p(AB), U_p(A)).$$

On the other hand, by (A.21), (A.22), and (A.18),

$$\sin \angle\big(U_1(\Lambda^p B), S_{\binom{d}{p}-1}(\Lambda^p A)\big) = \sin \angle\big(\iota(U_p(B)), \iota(S_{d-p}(A)^\perp)^\perp\big)$$

$$\geq [\sin \angle(U_p(B), S_{d-p}(A))]^{\min\{p, d-p\}}.$$

Combining the facts above we obtain the lemma. □

A.6. Expansion on the Grassmannian

The aim of this subsection is to prove the following lemma, which is used in Subsection 3.6.

Lemma A.10. *Given $\alpha > 0$, there exists $b > 0$ with the following properties. Let $A \in \mathrm{GL}(d, \mathbb{R})$. Suppose that $P \in \mathscr{G}_p(\mathbb{R}^d)$ and $Q \in \mathscr{G}_{d-p}(\mathbb{R}^d)$ satisfy*

$$\min\{\angle(P, Q), \angle(AP, AQ)\} \geq \alpha. \tag{A.23}$$

Then there exists $\delta > 0$ such that if $P_1, P_2 \in \mathscr{G}_p(\mathbb{R}^d)$ are δ-close to P then

$$d(AP_1, AP_2) \geq b\frac{\mathbf{m}(A|_Q)}{\|A|_P\|}d(P_1, P_2). \tag{A.24}$$

Before proving this lemma, we need still another characterization of the distance (A.5) on the Grassmannian. Suppose that $P, Q \in \mathscr{G}_p(\mathbb{R}^d)$ satisfy $d(P, Q) < 1$. Then $Q \cap P^\perp = \{0\}$, and so there exists a unique linear map

$$L_{Q,P}: P \to P^\perp \quad \text{such that} \quad Q = \{v + L_{Q,P}(v) : v \in P\}. \tag{A.25}$$

We have

$$\|L_{Q,P}\| = \frac{d(P, Q)}{\sqrt{1 - d(P, Q)^2}}; \tag{A.26}$$

indeed, letting $\theta \in [0, \pi/2]$ be such that $\sin \theta = d(P, Q)$, using (A.7) we conclude that $\|L_{Q,P}\| = \tan \theta$.

Lemma A.11. *Let P, P_1, $P_2 \in \mathscr{G}_p(\mathbb{R}^d)$. For each $i = 1, 2$, assume that $d(P_i, P) < 1/\sqrt{2}$, and let $L_i = L_{P_i, P}$. Then*

$$d(P_1, P_2) \leq \|L_1 - L_2\| \leq 4d(P_1, P_2).$$

Proof. Write any $u_1 \in P_1^\times$ as $u_1 = v + L_1 v$ for some $v \in P^\times$. By orthogonality, $\|u_1\| \geq \|v\|$. Letting $u_2 := v + L_2 v$, we have

$$\frac{\|u_1 - u_2\|}{\|u_1\|} \leq \frac{\|L_1 v - L_2 v\|}{\|v\|} \leq \|L_1 - L_2\|.$$

Using (A.7) we conclude that $d(P_1, P_2) \leq \|L_1 - L_2\|$.

For each $i = 1, 2$, we have $\|L_i\| \leq 1$, as a consequence of (A.26) and the hypothesis $d(P_i, P) < 1/\sqrt{2}$. Consider an arbitrary unit vector $v_1 \in P$. Let $w_1 := v_1 + L_1 v_1$, so $\|w_1\| \leq 2$. By (A.7), there exists $w_2 \in P_2^\times$ such that

$$\frac{\|w_2 - w_1\|}{\|w_1\|} \leq d(P_1, P_2).$$

Let $v_2 \in P$ be such that $w_2 = v_2 + L_2 v_2$. By orthogonality,

$$\|w_1 - w_2\| \geq \max\{\|v_1 - v_2\|, \|L_1 v_1 - L_2 v_2\|\},$$

so

$$\|L_1 v_1 - L_2 v_1\| \leq \|L_1 v_1 - L_2 v_2\| + \|L_2 v_2 - L_2 v_1\| \leq \|L_1 v_1 - L_2 v_2\| + \|v_2 - v_1\|$$
$$\leq 2\|w_1 - w_2\| \leq 2\|w_1\|d(P_1, P_2) \leq 4d(P_1, P_2).$$

Taking sup over unit vectors $v_1 \in P_1$ we obtain $\|L_1 - L_2\| \leq 4d(P_1, P_2)$. $\quad\square$

Proof of Lemma A.10. First consider the case $\alpha = \pi/2$, so (A.23) means that $Q = P^\perp$ and $AQ = (AP)^\perp$. Assume that P_1, P_2 are δ-close to P, for some small $\delta > 0$ to be chosen later. Recall notation (A.25) and, for each $i = 1, 2$, consider the linear maps $L_i := L_{P_i, P}$ and $M_i := L_{AP_i, AP}$, which are well-defined since $\delta < 1$ guarantees that $P_i \cap Q = \{0\}$. These maps are related by $L_i = (A^{-1}|_{AQ}) \circ M_i \circ (A|_P)$. As a consequence,

$$\|L_1 - L_2\| = \|(A^{-1}|_{AQ}) \circ (M_1 - M_2) \circ (A|_P)\| \leq \frac{\|A|_P\|}{\mathbf{m}(A|_Q)}\|M_1 - M_2\|.$$

Lemma A.11 entails

$$\|L_1 - L_2\| \geq d(P_1, P_2).$$

On the other hand, by (A.26) we have $\|L_i\| \leq d(P_i, P) < \delta$, and therefore $\|M_i\| \leq \|A^{-1}\| \|L_i\| \|A\| < 1$ provided δ is chosen sufficiently small (depending on A). Using (A.26) again we guarantee that $d(AP_i, AP) < 1/\sqrt{2}$. This allows us to apply Lemma A.11 and obtain

$$\|M_1 - M_2\| \geq \tfrac{1}{4}d(AP_1, AP_2).$$

Putting these three estimates together, we conclude that the lemma holds with $b = 1/4$ provided $\alpha = \pi/2$. The general case can be reduced to the previous one by changes of inner products, whose effect on all the quantities involved can be bounded by a factor depending only on α. $\quad\square$

A.7. Additional lemmas about dominated splittings

A.7.1. More about domination of sequences of matrices. Recall from §2.3 the definition of the sets $\mathcal{D}(K, p, \mu, c, I)$.

Lemma A.12. *Given $K > 1$ and $\mu, c > 0$, there exist $\ell \in \mathbb{N}$ and $\tilde{c} > c$ such that if $I \subset \mathbb{Z}$ is an interval and $(A_i)_{i \in I}$ is an element of $\mathcal{D}(K, p, \mu, c, I)$, then:*

(i) *If $n' < n < k$ all belong to I and $k - n \geq \ell$ then*

$$d\big(U_p(A_{k-1} \cdots A_{n+1} A_n), \ U_p(A_{k-1} \cdots A_{n'+1} A_{n'})\big) < \tilde{c} e^{-\mu(k-n)}.$$

(ii) *If $k < m < m'$ all belong to I and $m - k \geq \ell$ then*

$$d\big(S_{d-p}(A_{m-1} \cdots A_{k+1} A_k), \ S_{d-p}(A_{m'-1} \cdots A_{k+1} A_k)\big) < \tilde{c} e^{-\mu(m-k)}.$$

Proof. Given K, μ, and c, let $\ell \in \mathbb{N}$ be such that $c e^{-\mu\ell} < 1$. Fix $(A_n)_{n \in I} \in \mathcal{D}(K, p, \mu, c, I)$ and $k \in I$. If $n \in I$ satisfies $n \leq k - \ell$ then the space $P_n := U_p(A_{k-1} \cdots A_{n+1} A_n)$ is well-defined. If $n - 1 \in I$ then it follows from Lemma A.4 that

$$d(P_n, P_{n-1}) \leq K^2 c e^{-\mu(k-n)}.$$

Therefore, if $n' < n$ belongs to I then

$$d(P_n, P_{n'}) \leq d(P_n, P_{n-1}) + d(P_{n-1}, P_{n-2}) + \cdots + d(P_{n'+1}, P_{n'}) \leq \tilde{c} e^{-\mu(k-n)}$$

where $\tilde{c} := K^2 c / (1 - e^{-\mu})$. This proves (i). The argument applied to A^{-1} yields (ii). □

The following "extension lemma" is useful to deduce one-sided results from two-sided ones.

Lemma A.13. *Given $K > 1$ and $\mu, c > 0$, there exists $c' \geq c$ such that every one-sided sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(K, p, \mu, c, \mathbb{N})$ can be extended to a two-sided sequence $(A_n)_{n \in \mathbb{Z}}$ in $\mathcal{D}(K, p, \mu, c', \mathbb{Z})$.*

Proof. Fix $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(K, p, \mu, c, \mathbb{N})$. Let $Q_n := S_{d-p}(A_{n-1} A_{n-2} \cdots A_0)$, which is defined for sufficiently large n. By Lemma A.12, the spaces Q_n converge to some $Q \in \mathscr{G}_p(\mathbb{R}^d)$; moreover, we can find some n_0 depending only on the constants K, μ, c (and not on the sequence of matrices) such that for all $n \geq n_0$ we have $d(Q_n, Q) < 1/\sqrt{2}$ or equivalently, by (A.8), $\angle(Q_n, Q^\perp) < \pi/4$.

Let B be a matrix satisfying the following conditions:

$$\|B^{\pm 1}\| \leq K, \quad \frac{\sigma_{p+1}}{\sigma_p}(B) < e^{-\mu}, \quad B(Q) = Q = S_{d-p}(B), \quad B(Q^\perp) = Q^\perp = U_p(B).$$

In particular, for all $m \geq 0$ we have $\frac{\sigma_{p+1}}{\sigma_p}(B^m) < e^{-\mu m}$. Then, for all $n \geq n_0$, Lemma A.7 yields

$$\frac{\sigma_{p+1}}{\sigma_p}(A_{n-1} \cdots A_0 B^m) \leq 2c e^{-\mu(n+m)}.$$

As a consequence of Lemma A.2, a similar estimate holds for $0 \leq n < n_0$ with another constant replacing $2c$. So if we set $A_n = B$ for all $n < 0$ then the extended sequence $(A_n)_{n \in \mathbb{Z}}$ belongs to $\mathcal{D}(K, p, \mu, c', \mathbb{Z})$ for a suitable c' depending only on μ, c, and n_0. $\quad\square$

Combining the previous lemma with Proposition 2.4, we obtain:

Corollary A.14. *Given $K > 1$ and $\mu, c > 0$, there exist $\tilde{c}, \tilde{\mu}, \alpha > 0$ with the following properties. For every one-sided sequence $(A_n)_{n \in \mathbb{N}}$ in $\mathcal{D}(K, p, \mu, c, \mathbb{N})$, the limit*

$$Q = \lim_{n \to \infty} S_{d-p}(A_{n-1} \cdots A_0)$$

exist, and there exists $\tilde{Q} \in \mathscr{G}_p(\mathbb{R}^d)$ such that for every $n \geq 0$,

$$\angle\big(A_{n-1} \cdots A_0(\tilde{Q}),\, A_{n-1} \cdots A_0(Q)\big) \geq \alpha, \qquad \frac{\|A_{n-1} \cdots A_0|_Q\|}{\mathbf{m}(A_{n-1} \cdots A_0|_{\tilde{Q}})} < \tilde{c}e^{-\tilde{\mu}n}.$$

A.7.2. Hölder continuity of the bundles

Theorem A.15. *Let $\phi^t : X \to X$ a Lipschitz flow on a compact metric space X with $t \in \mathbb{T}$, and E a vector bundle over X. Let ψ^t be a β-Hölder linear flow over ϕ^t which admits a dominated splitting with constants $C, \lambda > 0$. If $\alpha < \beta$ and $e^{-\lambda}K^{\alpha} < 1$ where K is a Lipschitz constant for ϕ^1 and ϕ^{-1}, then the maps $x \mapsto E^{\mathrm{cu}}(x)$ and $x \mapsto E^{\mathrm{cs}}(x)$ are α-Hölder.*

Sketch of the proof. Choose Lipschitz approximations $\hat{E}^{\mathrm{cs}}$ and $\hat{E}^{\mathrm{cu}}$ of E^{cs} and E^{cu} respectively. One can define the bundle $\mathcal{E}$ over X corresponding to the linear maps from $\hat{E}^{\mathrm{cu}}(x)$ to $\hat{E}^{\mathrm{cs}}(x)$, that is, $\mathcal{E}(x) = \mathrm{Hom}(\hat{E}^{\mathrm{cu}}(x), \hat{E}^{\mathrm{cs}}(x))$. Let $\mathcal{T} \subset \mathcal{E}$ be given as $\mathcal{T} = \{(x, L) \in \mathcal{E} : \|L\| \leq 1\}$. We consider the standard graph transform $H : \mathcal{T} \to \mathcal{E}$ given by

$$H(x, L) = (\phi^1(x),\, H_x(L))$$

defined so that if $(w, v) \in \hat{E}^{\mathrm{cs}}(x) \oplus \hat{E}^{\mathrm{cu}}(x)$ is in the graph of L (i.e. $w = Lv$) then $(\psi^1 w, \psi^1 v)$ is in the graph of $H_x(L)$. It follows that the map H is β-Hölder and a standard computation shows that some iterate leaves the set $\mathcal{T}$ invariant.

Given two sections σ_0 and σ_1 of $\mathcal{E}$ such that $\sigma_i(x) \in \mathcal{T}$ for every $x \in X$, one shows that the α-Hölder distance between $H \circ \sigma_0$ and $H \circ \sigma_1$ is uniformly contracted[31] if $\alpha < \beta$ and $e^{-\lambda}k^{-\alpha} < 1$ where $k := \min_{x \neq y} \frac{d(\phi^1(x), \phi^1(y))}{d(x, y)}$. Indeed, the graphs are getting contracted at a rate similar to λ while points cannot approach faster than k, which shows that the α-Hölder distance contracts.[32]

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[31] To show that this metric is contracted, one needs to assume that the constant C appearing in the dominated splitting is 1. Otherwise, one can argue for an iterate and the same will hold.

[32] The need for $\alpha < \beta$ is evident, since the section cannot be more regular than the cocycle. In the computation this appears because an error term of the form $d(x, y)^{\beta - \alpha}$ appears which will then be negligible as $d(x, y) \to 0$ and gives the desired statement. See [CP, Section 4.4] for a similar computation.

As a consequence, there is a unique H-invariant section σ of this bundle, which is moreover α-Hölder.

It is direct to show that this section corresponds to the bundle E^{cu}. A symmetric argument shows that E^{cs} is α-Hölder, proving the result. $\qquad\square$

Corollary A.16. *If $u \mapsto \psi_u^t$ is a β-Hölder family of linear flows over a Lipschitz flow ϕ^t and ψ_0^t admits a dominated splitting, then there exists a neighborhood D of 0 such that the maps $(u, x) \mapsto E_u^{\mathrm{cs}}(x)$ and $(u, x) \mapsto E_u^{\mathrm{cu}}(x)$ are α-Hölder.*

Proof. Fix α as in the previous theorem. There exists a neighborhood D of 0 for which $\lambda K^\alpha < 1$ (here λ denotes the strength of the domination for ψ_u^t with $u \in D$).

Now one applies the previous theorem to the linear flow $\hat{\psi}^t$ over $\phi^t \times \mathrm{id} : X \times D \to X \times D$. Note that the Lipschitz constants of $\phi^1 \times \mathrm{id}$ and $\phi^{-1} \times \mathrm{id}$ are the same as the ones of ϕ^1 and ϕ^{-1}. The result follows. $\qquad\square$

References

[Ar]   Arnold, L.: Random Dynamical Systems. Springer Monogr. Math., Springer, Berlin (1998) Zbl 0906.34001 MR 1723992

[ABY]   Avila, A., Bochi, J., Yoccoz, J.-C.: Uniformly hyperbolic finite-valued $\mathrm{SL}(2, \mathbb{R})$-cocycles. Comment. Math. Helv. **85**, 813–884 (2010) Zbl 1201.37032 MR 2718140

[Be]   Benoist, Y.: Sous-groupes discrets des groupes de Lie. European Summer School in Group Theory, Luminy (1997)

[BM]   Blumenthal, A., Morris, I. D.: Characterization of dominated splittings for operator cocycles acting on Banach spaces. J. Differential Equations **267**, 3977–4013 (2019) Zbl 07063333 MR 3959484

[BG]   Bochi, J., Gourmelon, N.: Some characterizations of domination. Math. Z. **263**, 221–231 (2009) Zbl 1181.37032 MR 2506315

[Bog]   Bogopol'skiĭ, O.: Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent. Algebra and Logic **36**, 155–163 (1997) Zbl 0966.200021 MR 1485595

[BDP]   Bonatti, C., Díaz, L. J., Pujals, E.: A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. Ann. of Math. **158**, 355–418 (2003) Zbl 1049.37011 MR 2018925

[BDV]   Bonatti, C., Díaz, L. J., Viana, M.: Dynamics beyond Uniform Hyperbolicity: a Global Geometric and Probabilistic Perspective. Encyclopaedia Math. Sci. 102, Mathematical Physics III, Springer, Berlin (2005) Zbl 1060.37020 MR 2105774

[Bow]   Bowditch, B. H.: A topological characterization of hyperbolic groups. J. Amer. Math. Soc. **11**, 643–667 (1998) Zbl 0906.20022 MR 1602069

[BCLS] Bridgeman, M., Canary, R., Labourie, F., Sambarino, A.: The pressure metric for Anosov representations. Geom. Funct. Anal. **25**, 1089–1179 (2015) Zbl 1360.37078 MR 3385630

[BCS] Bridgeman, M., Canary, R., Sambarino, A.: An introduction to pressure metrics on higher Teichmüller spaces. Ergodic Theory Dynam. Systems **38**, 2001–2035 (2018) Zbl 1397.37032 MR 3833339

[BH] Bridson, M. R., Haefliger, A.: Metric Spaces of Non-Positive Curvature. Grundlehren Math. Wiss. 319, Springer, Berlin (1999) Zbl 0988.53001 MR 1744486

[Ca] Cannon, J. W.: The combinatorial structure of cocompact discrete hyperbolic groups. Geom. Dedicata **16**, 123–148 (1984) Zbl 0606.57003 MR 0758901

[Ch] Champetier, C.: Petite simplification dans les groupes hyperboliques. Ann. Fac. Sci. Toulouse Math. (6) **3**, 161–221 (1994) Zbl 0803.53026 MR 1283206

[CK] Colonius, F., Kliemann, W.: The Dynamics of Control. Birkhäuser Boston, Boston, MA (2000) Zbl 0998.93502 MR 1752730

[CDP] Coornaert, M., Delzant, T., Papadopoulos, A.: Géométrie et théorie des groupes: Les groupes hyperboliques de Gromov. Lecture Notes in Math. 1441, Springer, Berlin (1990) Zbl 0727.20018 MR 1075994

[CP] Crovisier, S., Potrie, R.: Introduction to Partially Hyperbolic Dynamics. School on Dynamical Systems, ICTP, Trieste (2015)

[DT] Deroin, B., Tholozan, N.: Dominating surface group representations by Fuchsian ones. Int. Math. Res. Notices **2016**, 4145–4166 Zbl 1404.53046 MR 35446332

[Ebe] Eberlein, P.: Geometry of Nonpositively Curved Manifolds. Chicago Lectures in Mathematics, Univ. of Chicago Press (1996) Zbl 0883.53003 MR 1441541

[EHY] Eliasson, H., Hofer, H., Yoccoz, J.-C. (org.): Dynamische Systeme. Abstracts from the workshop held July 19–25, 2015. Oberwolfach Rep. **12**, 1937–1988 (2015) Zbl 1380.00070

[Ga] Gantmacher, F. R.: The Theory of Matrices. Vol. 1. Reprint of the 1959 translation, AMS Chelsea Publ., Providence, RI (1998) Zbl 0927.15001 MR 1657129

[GH] Ghys, E., de la Harpe, P. (eds.): Sur les groupes hyperboliques d'après Mikhael Gromov. Papers from the Swiss Seminar on Hyperbolic Groups held in Bern, 1988. Progr. Math. 83, Birkhäuser Boston, Boston, MA (1990) Zbl 0731.20025 MR 1086648

[Go] Gogolev, A.: Diffeomorphisms Hölder conjugate to Anosov diffeomorphisms. Ergodic Theory Dynam. Systems **30**, 441–456 (2010) Zbl 1185.37040 MR 2599887

[Gr] Gromov, M.: Hyperbolic groups. In: Essays in Group Theory, MSRI Publ. 8, Springer, 75–263 (1987) Zbl 0634.20015 MR 0919829

[GGKW] Guéritaud, F., Guichard, O., Kassel, F., Wienhard, A.: Anosov representations and proper actions. Geom. Topol. **21**, 485–584 (2017) Zbl 1373.37095 MR 3608719

[GKW] Guéritaud, F., Kassel, F., Wolff, M.: Compact anti-de Sitter 3-manifolds and folded hyperbolic structures on surfaces. Pacific J. Math. **275**, 325–359 (2015) Zbl 1346.57005 MR 3347373

[GW] Guichard, O., Wienhard, A.: Anosov representations: domains of discontinuity and applications. Invent. Math. **190**, 357–438 (2012) Zbl 1270.20049 MR 2981818

[GJT] Guivarc'h, Y., Ji, L., Taylor, J. C.: Compactifications of Symmetric Spaces. Progr. Math. 156, Birkhäuser, Boston, MA (1998) Zbl 1053.31006 MR 1633171

[Hel] Helgason, S.: Differential Geometry and Symmetric Spaces. Academic Press, New York (1962) Zbl 0111.18101 MR 0145455

[HPS] Hirsch, M., Pugh, C., Shub, M.: Invariant Manifolds. Lecture Notes in Math. 583, Springer, Berlin (1977) Zbl 0355.58009 MR 0501173

[Hit]   Hitchin, N. J.: Lie groups and Teichmüller space. Topology **31**, 449–473 (1992) Zbl 0769.32008   MR 1174252

[Hö]   Hörmander, L.: An Introduction to Complex Analysis in Several Variables. 3rd ed., North-Holland, Amsterdam (1990)   Zbl 0685.32001   MR 1045639

[HJ]   Horn, R. A., Johnson, C. R.: Matrix Analysis. 2nd ed., Cambridge Univ. Press, Cambridge (2013)   Zbl 1267.15001   MR 2978290

[Hub]   Hubbard, J.: Teichmüller Theory and Application to Geometry, Topology, and Dynamics. Vol. 1. Matrix Editions, Ithaca, NY (2006)   Zbl 1102.30001   MR 2245223

[Hum$_1$] Humphreys, J.: Introduction to Lie Algebras and Representation Theory. 2nd printing revised, Grad. Texts in Math. 9, Springer, New York (1978)   Zbl 0447.17002   MR 0499562

[Hum$_2$] Humphreys, J.: Linear Algebraic Groups. Grad. Texts in Math. 21, Springer, New York (1975)   Zbl 0471.20029   MR 0396773

[Kap]   Kapovich, M.: Representations of polygons of finite groups. Geom. Topol. **9**, 1915–1951 (2005)   Zbl 1163.20029   MR 2175160

[KLP$_1$] Kapovich, M., Leeb, B., Porti, J.: Morse actions of discrete groups on symmetric space. arXiv:1403.7671 (2014)

[KLP$_2$] Kapovich, M., Leeb, B., Porti, J.: A Morse Lemma for quasigeodesics in symmetric spaces and euclidean buildings. Geom. Topol. **22**, 3827–3923 (2018)   Zbl 06997379   MR 3890767

[KLP$_3$] Kapovich, M., Leeb, B., Porti, J.: Some recent results on Anosov representations. Transform. Groups **21**, 1105–1121 (2016)   Zbl 1375.37131   MR 3569569

[Lab]   Labourie, F.: Anosov flows, surface groups and curves in projective space. Invent. Math. **165**, 51–114 (2006)   Zbl 1103.32007   MR 2221137

[Lang] Lang, S.: Fundamentals of Differential Geometry. Grad. Texts in Math. 191, Springer, New York (1999)   Zbl 0932.53001   MR 1666820

[Led]   Ledrappier, F.: Structure au bord des variétés à courbure négative. In: Séminaire de Théorie Spectrale et Géométrie 13, Univ. Grenoble I, Saint-Martin-d'Hères, 97–122 (1995)   Zbl 0931.53005   MR 1715960

[Len]   Lenz, D.: Existence of non-uniform cocycles on uniquely ergodic systems. Ann. Inst. H. Poincaré Probab. Statist. **40**, 197–206 (2004)   Zbl 1042.37002   MR 2044815

[LM]   Lind, D., Marcus, R.: An Introduction to Symbolic Dynamics and Coding. Cambridge Univ. Press, Cambridge (1995)   Zbl 1106.37301   MR 1369092

[Mat]   Mathéus, F.: Flot géodésique et groupes hyperboliques d'après M. Gromov. In: Séminaire de Théorie Spectrale et Géométrie 9, Univ. Grenoble I, Saint-Martin-d'Hères, 67–87 (1991)   Zbl 0747.53036   MR 1715930

[Mi]   Mineyev, I.: Flows and joins of metric spaces. Geom. Topol. **9**, 403–482 (2005)   Zbl 1137.37314   MR 2140987

[Pa]   Palmer, K. J.: Exponential separation, exponential dichotomy and spectral theory for linear systems of ordinary differential equations. J. Differential Equations **46**, 324–345 (1982)   Zbl 0466.34025   MR 0681227

[PS]   Potrie, R., Sambarino, A.: Eigenvalues and entropy of a Hitchin representation. Invent. Math. **209**, 885–925 (2017)   Zbl 1380.30032   MR 3681396

[Ru]   Ruelle, D.: Analyticity properties of the characteristic exponents of random matrix products. Adv. Math. **32**, 68–80 (1979)   Zbl 0426.58018   MR 0534172

[Sa]   Sambarino, A.: The orbital counting problem for hyperconvex representations. Ann. Inst. Fourier (Grenoble) **65**, 1755–1797 (2015)   Zbl 1354.22016   MR 3449196

[Sam]   Sambarino, M.: A (short) survey on dominated splittings. In: Mathematical Congress of the Americas, Contemp. Math. 656, Amer. Math. Soc., Providence, RI, 149–183 (2016) Zbl 1378.37066 MR 3457601

[Sim]   Simon, B.: Harmonic Analysis. A Comprehensive Course in Analysis, Part 3. Amer. Math. Soc. (2015) Zbl 1334.00002 MR 3410783

[Sma]   Smale, S.: Differentiable dynamical systems. Bull. Amer. Math. Soc. **73**, 747–817 (1967) Zbl 0202.55202 MR 0228014

[Ste]   Stewart, G. W.: Matrix Algorithms. Vol. I: Basic Decompositions. SIAM, Philadelphia, PA (1998) Zbl 0910.65012 MR 1653546

[Ti]   Tits, J.: Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. J. Reine Angew. Math. **247**, 196–220 (1971) Zbl 0227.20015 MR 0277536

[Wie]   Wienhard, A.: Representations and geometric structures. arXiv:1602.03480 (2016)

[Wo]   Wong, Y.-C.: Differential geometry of Grassmann manifolds. Proc. Nat. Acad. Sci. U.S.A. **57**, 589–594 (1967) Zbl 0154.21404 MR 0216433

[Yo]   Yoccoz, J.-C.: Some questions and remarks about SL(2, $\mathbb{R}$) cocycles. In: Modern Dynamical Systems and Applications, Cambridge Univ. Press, Cambridge, 447–458 (2004) Zbl 1148.37306 MR 2093316